THE ARITHMETIC TEACHER

Volume IV **Number 4**

October **1957**

The Gifted Ones—How Shall We Know Them?

Charlotte Junge
Wayne University, Detroit

A few years ago a teacher could be reasonably certain *that the majority of the students in his classroom were capable of maintaining* the minimum level of achievement suggested by the curriculum. Today the situation is very different. Universal education brings together *all the children of all people.* They represent *widely different capacities and interests* which cannot be satisfied through uniform content and method. Perhaps one of the most important steps toward understanding the modern school is to realize the *range of ability with which the teacher must work. Research* indicates that when the *average group of six-year olds enter school that two per cent* of them will be *below the average for four-year olds in general* mental development and two per cent *will be above the average for eight-year olds.* Considering only the middle *96% of the class there is still a four year* range in general intelligence. This same range of ability will be found in achievement in any particular area. Furthermore, this range in achievement tend to increase as the children progress through school. A typical range of ability in arithmetic is shown by the following data.[1]

(Scores on Arithmetic test of the Iowa Every Pupil Test of Basic Skills)

Grade	3	4	5	6
Hi-Score	5.9	5.9	8.0	8.9
Mid-score	3.9	4.8	5.7	7.2
Lo-score	2.8	3.3	4.1	4.7

This situation creates many problems where children are taught en masse. Rapid learners are allowed to coast through the grades with a minimum of challenge to their talents while slow learners are frustrated by tasks which demand more than they are capable of achieving. Educators agree that every child should be challenged to reach his maximum of intellectual, social and emotional development. Teachers and administrators are steadily confronted therefore, with finding materials and methods of teaching which will make maximum allowance for individual differences. It is the purpose of this discussion to suggest the means by which the able learner can be recognized so that adjustments in method and curriculum can be made to meet his needs.

At school age, the slow learner, the average learner, and the fast learner look much alike, especially to the elementary teacher who attempts to instruct a group of 30 to 40 pupils each day. The first responsibility we have then, in planning work for the gifted child is to recognize him and to evaluate him in terms of his arithmetic achievements.

Characteristics of Gifted Children

Prior to 1920, it was commonly believed that very bright "gifted" children were

[1] Spitzer, H. F., *The Teaching of Arithmetic,* Houghton Mifflin, 1954 Rev. Ed., p. 400.

atypical, immature and emotionally unstable. Some writers observed that eccentricity and genius were inseparable—while others stated that the extent of genius was in direct proportion to the degree of instability!

Contrary to this belief, recent studies and reports[2] indicate that the gifted child is physically stronger, socially more secure, and emotionally more stable than the average child of his own chronological age. In addition, he is more alert and responsive and more eager to learn. A composite list of the traits most frequently found in gifted students includes the following:—

1. high verbal comprehension
2. superior vocabulary
3. intellectual curiosity and imagination
4. the ability to assimilate and generalize
5. objective self-analysis
6. persistency
7. insight

The identification of the "gifted" child in arithmetic is not as easy as it appears at first thought. The use of I.Q. as the sole criterion brings about selection of many children who possess high abstract intelligence. However, for children reared under limiting socio-economic or poor educational circumstances the verbal test of intelligence has limited value in determining their potential.

In a recent talk before this council, Marguerite Brydegaard[3] has suggested that we are often "Penny Wise and Pound Foolish" if we evaluate the performance of children in arithmetic *only* in terms of ability to perform computations and give prompt answers to arithmetical facts. A child who performs well on tests on computation is *not necessarily the bright child* in arithmetic.

Similarly, performance on achievement tests cannot be used as a sole criterion for selection of the able learner. The lack of interest in classroom routine, the attitudes of frustration built up because of insufficient challenge may cause bright children to perform poorly on achievement tests and to be judged dull and slow-learning by the teacher.

Since the rapid learner constitutes somewhere in the neighborhood of 3% of the population, we must start early in the school career to identify these children. They *can be identified sometimes* by preformance on intelligence and achievement tests, they can, at times, be identified by facile, accurate work in computation, they *may be* the children who finish their work first. If in addition to these things, a child is also excellent in mathematical understanding, in the ability to generalize, to interpret, and apply mathematical concepts then he should in all probability be classified as one of the brighter students in arithmetic.

Facility in thinking and superior mental organization are qualities of the gifted child which are recognized by all investigators. In identifying the bright child in arithmetic, good teachers will observe the work habits and work products of each child carefully. She will note the way he responds to new learning, the readiness with which he grasps basic understandings, his skill in extending present learning into new, but similar situations.

Specifically she will note that *he learns more rapidly than the average child, that he moves quickly into abstract thinking and works frequently at the level of insight in solving quantitative situations.* Bob, a five-year old in kindergarten, demonstrated this ability in helping Ted solve a "construction" problem. Ted announced that he couldn't finish his building because he needed a long block and all long blocks were gone. Bob surveyed the situation and said, "Get two short blocks Ted, it makes a long block."

Similarly, Tommy in grade three who was being introduced to the idea of division moved from the real problem situation to abstraction easily when faced with the problem "Mary has eight cookies, she want to share them equally with four of her friends.

[2] Paul Witty, *The Gifted Child.* Boston: D. C. Heath and Co. 1951.

[3] Marguerite Brydegaard, "Creative Ways of Teaching," THE ARITHMETIC TEACHER, February 1954. pp. 21—24.

how many will she give to each?" Tommy quickly drew four round "faces" and below each wrote the numeral 2.

○ ○ ○ ○
2 2 2 2

The other children in the class carefully drew four circles and put two marks in each circle and then counted to find the number each received. By contrast, Tommy had moved mentally through this stage and simply recorded the numeral two for each child.

Or, consider ten-year-old Mary, who, being quizzed by her father to see if she understood column addition, was asked to add $29+28+27$. Without touching pencil to paper she thought for a moment and announced that it would be "90 less 6" or 84!

As the teacher carefully observes and listens to children she will notice that the bright child has the ability to generalize easily, to recognize relationships, to comprehend meanings and to think logically. This will be the child who deduces that 3×2 tens will be 6 tens because $3\times2=6$; he will be the one who will know that 3.5 multiplied by .4 will be less than 2 because .4 is less than one half; this is the child who will know that if $30\times16=480$ then 15×16 will be half as much; he will be able to picture relationships and to express ideas in diagrams and will be among the first in the class to state principles of operation from these diagrams.

The bright child in arithmetic will be observed to differ from his classmates in mental habits and ways of working. He will be less patient with routine procedures, he is apt to be rebellious with assignments which require meticulous detail, and non-conformist in his methods of solving problems. This is the child who will try to solve problems in a variety of ways, who will rebel at putting all steps of solutions on paper, and who will think of new situations in which to apply his learning. Let us realize that this child needs to learn efficient methods of study, but let us also know that he will be able to work better unhampered by too close supervision. His is a creative mind which needs guidance and direction but one which should not be channeled into set patterns of thinking.

Another characteristic of the gifted child in arithmetic will be this ability to "see" in advance a series of steps, and to *sense the direction* in which one must go to solve the problem. This the average child cannot do, they are the "one-step" minds, which see only one relation at a time. The creative thinker sees far ahead, carried in his mind a series of related things to do. This is experimental thinking with an end in view. This is "intellectual map-making." The child who thinks in this way has little difficulty seeing through problem situations. He is quick to sense cause and effect relationships and to plan a course of action accordingly. He may say "If I do that, this will follow." He can give clear, logical explanations for the steps he follows in solving a problem when pressed to do so—but in actual practice probably moves so quickly from step to step that he may be unaware of the steps he takes.

The child who thinks in this way continually re-appraises the direction of his thought—he is quick to sense a wrong cue, to note errors, and to be aware of discrepancies. He shows "common sense" in safe-guarding his thinking: estimation and approximation are second nature to him. He uses them steadily in checking on his thinking, and eventually takes the step from an estimated answer to exact mental computation, without reference to pencil and paper. For example, a nine-year-old in finding the answer to 4×89 cents might first think, it will be about \$3.60—*and then*—"It will be 4 cents less than \$3.60, it will be \$3.56." Or a sixth grade child obtaining an answer of 28 to $7488\div36$, would quickly re-appraise his work saying "My answer can't possibly be twenty-eight since there are more than 100 36's in 7488. My answer must have three places."

Curiosity Is a Clue

Along with the characteristics given above, the bright child in arithmetic evidences a high-powered, intellectual curiosity which leads him to search out facts for himself, to set arithmetical tasks for himself and to be aware of the quantitative aspects of his home and school environment. He shows interest in investigating topics in encyclopedias and general reference books; he brings in problems found in newspapers and magazines, and he makes up problems from data in science and social studies. He thoroughly enjoys working with numbers and is confident and self-reliant. He is a "continuous learner" ever extending his present learning into new fields.

There is no single method one may use to determine these "special children." We must use all ways which have stood the test of time; standard tests of intelligence and achievement should be employed; informal teacher made tests and inventories should be part of the identification procedure; and nothing can replace the careful, clinical observation of good teachers as they work with children day by day. We must not only be careful and thorough in our search for these children, but we *must begin early*—kindergarten and first grade is not too soon, for many children come to school with a well developed interest in numbers and an amazingly good grasp of number usage. A study, only recently completed by Corwin Bjonerud at Wayne State University shows that today's four and five-years olds have a far better understanding of numbers than most educators realize. One four-year old, shown a group of objects for a brief period and asked, "how many?" said "I saw four (he paused) and three and two more, I saw nine!" Another child of similar age, given the following problem: May had 8 letters. She mailed two of them. How many does she have left to mail? She promptly answered 6. Obviously these two youngsters have some of the characteristics and interests which would indicate special abilities in arithmetic—whether or not these abilities will continue to develop will depend on the quality of the teaching they are given and the quality of the learning experiences in the curriculum.

The Teacher's Responsibility

Once the gifted children are identified the task of the school and the teacher becomes clear. We must realize that "the gifted child" is both an asset and a responsibility. He is an asset of incalculable value to society. His potentialities for good are difficult to over estimate. Our socio-economic structure demands leadership of the highest quality and keenest intelligence."[4] Today perhaps as never before, education is being challenged to develop leadership for the tremendous tasks which lie ahead. Under such conditions special attention to the intellectually gifted in our schools is not only justified, but demanded and plans must be made for initiating an enriched program to challenge the abilities of these children. In general, it is agreed, that there will be little, if *any differentiation of topics in arithmetic for children of different abilities.* There is a *core of understanding* and *skills* needed by all children. *Differentiation and enrichment*, will therefore be *not in topics*, but in *levels of learning, in depth, and in scope.* The amount of concrete background in any topic can be varied, rates of learning can be varied, and extent of a topic can be varied. Teaching so as to make variations in depth and scope is made easier by grouping. However, *grouping* in arithmetic is carried on in a fashion different from that in reading. In reading learning is carried on in a "ragged front." In a fourth grade, for example, one group of children will be reading from fourth grade materials, another may be reading third grade materials and still another group may be reading from source materials rated fifth or sixth grade in reading level. However in arithmetic, the learning is carried on in an "even front." All pupils in class work on the same area or concept but distinct provision is made for different levels of work

[4] Natl. Society for the Study of Education, *Forth-ninth* Yearbook, Part II, *The Education of Exceptional Children.* 1950, p. 260.

in the area. The provision for different levels of work within the same area makes possible the retention of class unity at the same time it encourages children to proceed at the level of understanding suited to their ability. For example, a fifth grade class might all be working on the addition of *like fractions*. Some of the children in the group will be working with manipulative materials under the guidance of the teacher. A second group will be working more or less independently using manipulative materials but also recording solutions abstractly. A third group of able learners will be working independently, solving problems involving the understanding and finding ways of proving the truth of their solutions. All children are working on developing an understanding of the same concept, but the able child is working at a level which challenges his thinking.

Learning for the Gifted

Working within this general framewok of grouping in terms of levels of thinking, the elementary teacher will find opportunity for providing learning experiences of the following types which will stimulate the able learner to *do creative thinking*. She will:

1. *Encourage the bright child to use more mature methods* of thinking and solving problems. She will require them to *diagram* and *show proof* of solutions. She will encourage them to experiment and to *solve* problems in a *variety of ways*.
2. *Help children find various ways of checking problems*. For example, the casting out of nines, the reducing to a single digit, or the use of the civil service check in addition.
3. *Provide opportunities for children to investigate* topics in encyclopedias and general reference books. They might investigate:
   (1) ways time has been told down thru the ages
   (2) number systems used in different civilizations
   (3) old ways of multiplying and dividing, etc.
   (4) the history of measures and the development of new measures
   (5) local rates for water, electricity, and gas
   (6) safety instruments used on airplanes
   (7) why ice floats
   (8) how many miles an hour a spot on the equator moves
   (9) the amount of water in some of our fruits and vegetables
   (10) how the speed of the wind is measured
   (11) what information is given on a weather map
   (12) the sizes of automobile tires
   (13) the scales to which maps are grown
   (14) how to locate places on a road map
   (15) how to find the distance around a bicycle tire
   (16) how the weatherman uses the barometer
   (17) decimals used in reference books
   (18) the Dewey decimal system used in libraries
   (19) the history of decimal fractions as given in a reference book
   (20) decimal fractions found in newspapers
   (21) decimal fractions in reference books and and in text books of other fields of study
   (22) weather reports in newspapers
   (23) how people use decimal fractions in industry
   (24) the *mill* as a unit of money
   (25) the use of decimal fractions in athletics
   (26) sales slips used in local stores
   (27) how to open a charge account
   (28) wages paid in building trades
   (29) why people are given credit in stores
   (30) the history of the United States census
   (31) record flights by airplanes
   (32) measuring devices found in automobiles
   (33) everyday uses of division
   (34) portraits on our paper money
   (35) foreign money
4. *Provide problems which require first hand research through gathering, interpreting, and analyzing data.*
5. *Use actual problem situations* arising in home or school and require children to play ways of solving them.
6. *Encourage them to solve problems without pencil and paper*. The skill of estimating and working with rounded numbers should be stressed throughout the arithmetic program.
7. *Provide opportunities for them to consult* maps, charts, graphs, tables and diagrams and to solve problems from these materials. Older children can be helped to show arithmetical data in tabular and graphic form.
8. *Give special vocabulary exercises* in which children explain or illustrate meanings of arithmetical terms.
   (1) The words in the second column, below, mean the same as the groups of words in the first column. Read each group of words in the first column and find the word in the second column that means the same.

I

(a) The distance around a square of rectangle
(b) The answer in division
(c) The answer in subtraction
(d) The number we divide by
(e) The number we multiply by
(f) The answer in addition
(g) The answer in multiplication
(h) A short way to write the name of a measure
(i) What a thermometer shows
(j) A remainder in division
(k) Morning
(l) Afternoon

II

P.M.	quotient
product	R
add	multiplier
temperature	divisor
A.M.	multiply
perimeter	sum
subtract	abbreviation
remainder	

(2) Explain the meanings of the words below:

account	place value
average	price
billion	product
budget	quantity
difference	quotient
budget	round off
denominator	sum
divident	quotient
division	remainder
estimate	round number
fraction	sum
million	unit
ounce	whole
estimate	tens' place
numerator	terms
product	value

9. *Relate the work in arithmetic* to learnings in other areas, particularly in science and in social studies. Arithmetical ideas included in the content areas should be analyzed and interpreted. The able learner can do this for the others in the class.
10. *Use number tricks and puzzles* to challenge thinking and to increase interest in arithmetic. All children should develop feelings of confidence about arithmetic and to know that it can "be fun." A recent book by Herbert Spitzer provides many splendid activities of this sort for teachers in grades two and above. (*Practical Classroom Procedures for Enriching Arithmetic*, Houghton Mifflin, 1956.)
11. *Finally, good teaching in arithmetic helps* children develop a method of attack upon problem situations. It will develop individuality and independence and will help children evaluate their own learning procedures.

Summary

It has been the purpose of this paper to suggest that the way of identifying individual differences in arithmetic and to recommend procedures of flexible grouping which recognizes various levels of thinking and which varies, not in content, but in terms of depth and scope. The more able pupils will move quickly to levels of abstraction and will have opportunities to extend and apply the understandings developed in new and different situations. The able learner, in such a situation, will be stimulated to creative thinking and will develop a healthy self-reliance in solving quantitative situations.

EDITOR'S NOTE. Dr. Junge not only helps us to identify the gifted pupils in our schools but she also tells us some of the things we should do with them in arithmetic. She points out that identification is not as simple as using an ordinary intelligence test. She has given the characteristic "earmarks" of these youngsters and shows how, through observation and interview, we may know them. Gifted pupils are not all alike, they too have individual differences in both kind and amount. We need to understand them and to encourage and aid their development. Teachers are often amazed when they discover what some of their able pupils are learning and doing independently outside the school. A few words of encouragement, an expression of interest, a leading question, a suggestion of an avenue of investigation, or a hint toward redirection may be all that is needed to stimulate certain pupils. Gifted youngsters deserve the type of teaching which guides and stimulates independent discovery and learning. With these pupils, the mode of learning is probably more important than the mathematical content. Dr. Junge has pointed the way, let us explore the route.

Some Questionable Arithmetical Practices

(*Continued from page 178*)

EDITOR'S NOTE. Professor Spitzer is worried about our careless use of words and ideas in arithmetic. He points out that things such as actions cannot easily be pictured in textbooks. The stages from beginning with the manipulation of objects which illustrate or represent quantities to the more final use of abstract symbols are difficult to reproduce in print. This may be an argument for less reliance upon textbooks at certain stages of learning. He would like teachers and books to use language and definitions that are precise and correct. Perhaps he has had pupils who have pointed out that many book statements do not say exactly what is meant. We do have a problem in stating things so that a child can understand them and at the same time have a statement which an adult will respect. Perhaps this cannot be done. However, a teacher should constantly seek both clarity and precision at whatever stage she is working with pupils.

Developing Ability in Mental Arithmetic

FRANCES FLOURNOY
University of Texas, Austin

IT IS A MATTER OF COMMON EXPERIENCE and observation that life presents many uses for mental arithmetic in arriving at quick solutions to arithmetical situations. Paper and pencil should seldom be necessary for interpreting many of these quantitative situations. Because activities of everyday life require competence in mental arithmetic, schools must provide pupils with opportunity to learn to think without paper and pencil in solving problems involving simple computation, making approximations, and interpreting quantitative data, terms, and statements.

Historical Background

Mental arithmetic has received little attention in our schools in the last fifty years. The swing toward little or no emphasis on mental arithmetic about the beginning of the twentieth century seems to have arisen because of an over-emphasis on mental arithmetic during the latter part of the nineteenth century. In presenting a brief history of mental arithmetic, Smith[1] has stated:

> About the middle of the last century mental arithmetic underwent a great revival largely through the influence of Pestolozzi in Europe and Warren Colburn in this country, in each case as a protest against the intellectual sluggishness, lack of reasoning, and slowness of operation of the old written arithmetic. For a long time the oral form was emphasized, in America doubtless unduly so; and this was naturally followed by such a reaction that it lost practically all of its standing.

Though mental arithmetic had lost practically all of its standing in American education by the beginning of the twentieth century, a few prominent thinkers in the field of education continued to point out the need for mental as well as written exercises in arithmetic. In relation to this problem, Thorndike[2] asserted:

> The common practice of either having no use made of paper and pencil or having all computations and even much verbal analysis written out elaborately for examination is favorable for learning. The demands which life itself will make of arithmetical knowledge and skill will range from tasks done with every percentage of written work from zero up to the case where every main result obtained by thought is recorded for later use by further thought.

Smith[3] has made the following observation concerning a need for providing instruction in mental arithmetic in our schools:

> The ordinary purchase of household supplies requires a practical ability in the mental arithmetic of daily life, and this ability comes to mind only through repeated exercise. It is a fair inference from statistical investigations that a person may be rapid and accurate in written work but slow and uncertain in oral solutions. Therefore it will not do, from a practical standpoint, to drill children only in written arithmetic if we expect them to be reasonably ready in purely mental work.

Within the last ten years there has been a gradual growth in the amount of mental arithmetic exercises included in arithmetic text books for children. This seems to indicate that authorities in the field of arithmetic are beginning to realize that written arithmetic alone does not adequately prepare the child to meet without paper and pencil the arithmetical situations of life. It is the concern of this writer, however, that present textbooks in elementary school arithmetic still do not provide systematic attention to developing mental arithmetic ability and do not provide sufficiently varied experiences with mental arithmetic to meet the varied needs for it in everyday situations.

[1] David Eugene Smith "Mental Arithmetic." *A Cyclopedia of Education*, Vol. IV, Macmillan Company, New York, 1918, pp. 195–196.

[2] Edward L. Thorndike, *The Psychology of Arithmetic*, Macmillan Company, New York, 1922, pp. 263–264.

[3] David Eugene Smith. *The Teaching of Arithmetic*, Ginn and Company, Boston, 1913, p. 55.

Kinds of Situations Requiring Mental Arithmetic Ability

Life's needs for mental arithmetic may be organized into four basic situations:

1. A problem situation in which one needs to arrive at an exact answer—

   Jimmy is standing at the valentine counter in a variety store. He sees that some of the valentines are 3¢ each and some are 5¢ each. He wants to buy valentines for 30 friends and has $1.40 to spend. Jimmy likes the 5¢ valentines better, but he is wondering how may 5¢ valentines he can buy for $1.40. Will he have to buy the 3¢ valentines?

   In this situation Jimmy must be able to do several things:

   (a) Recognize his problem and organize the facts of the problem.

   (b) Keep the numbers in mind as he thinks about the problem.

   (c) Perform the necessary arithmetical process or processes, without paper and pencil, and reach a decision about his problem.

2. A problem situation in which one needs only to arrive at an approximate answer—

   While resting alter a swim, Bill and Sam are talking about a week-end trip they will make with their families to see their grandmothers. Bill says, "Our round trip will be 88 miles. We have made the trip before and we spend about 2 hours in total travel time."

   Sam says, "Our round trip will be 178 miles. This is our first trip since we moved. If we travel about the same rate you travel, I wonder about how much total travel time the trip will require."

   In this situation Bill and Sam must be able to do several things:

   (a) Recognize the problem and organize the facts of the problem.

   (b) Keep these facts in mind as they think about the problem.

   (c) Round the larger numbers in the problem.

   (d) Perform the necessary process or processes with the rounded numbers, without paper and pencil, and arrive at an approximate answer.

3. Interpreting quantitative terms and statements, as such quantities are heard or read, in terms of a familiar "reference measure." As used here "reference measure" refers to a familiar quantity within a person's experience and with which the unfamiliar quantity may be compared.

   Mary is giving a social studies report. She has just told the class that the Empire State Building in New York City is 1250 feet high. Her teacher asks whether anyone in the class has ever seen the Empire State Building. Alice says that she has seen the building and that this is very tall.

   In order to encourage the class to try to visualize better this height, Mary's teacher asks with what tall building in their city the Empire State Building might be compared. Jack says, "My Dad told me that the new building downtown is 300 feet high."

   Since everyone in the class has seen the new building, the teacher asks, "How many buildings, the height of the new building, if stacked one on the other, would it take to equal the height of the Empire State Building?"

   In such a situation the class must be able to do several things:

   (a) Select a familiar height as a "reference measure."

   (b) Round the numbers mentioned as needed

   (c) Perform the correct process for making the comparison. Then try to understand better the height referred to in the report.

4. Reading and using tables, graphs, and scales found in the context of encyclopedias, newspapers, and factual social studies and science books.

Textbook Exercises Needed to Develop Mental Arithmetic Ability

Arithmetic textbooks should provide mental arithmetic experiences that are well-balanced as to amount and types. Textbooks might well provide mental arithmetic exercises of the following types:

1. Learning short-cuts for adding, subtracting, multiplying, and dividing small numbers without the aid of paper and pencil.
2. Practice in computing for exact answers without the aid of paper and pencil.
3. Practice in solving word problems with simple numbers, for exact answers without paper and pencil, read from the textbook by the pupil.
4. Practice in solving word problems with simple numbers, for exact answers without paper and pencil, as the pupil listens to the teacher read it but does not see it in print.
5. Constructing problems orally for other members of the class to listen to and solve without the aid of paper and pencil.
6. Learning reasons for and advantages in using rounded numbers.
7. Learning to judge when to use rounded numbers and when to use exact numbers.
8. Practice in rounding numbers.
9. Practice in estimating answers when adding, subtracting, multiplying, and dividing.
10. Practice in estimating answers to word problems.
11. Realizing the importance of properly interpreting quantitative terms and statements found in reading reference materials.
12. Experience in selecting familiar "reference measures" and learning to use these in interpreting unfamiliar measures.
13. Understanding the importance of tables, graphs, and scales found in reading reference materials.
14. Practice in reading and using tables, graphs, and scales.

Analysis of Types of Mental Arithmetic Exercises Included in Textbooks

Table I presents the results of an examination of six fifth-grade arithmetic textbooks published in the last five years. These books were examined for the purpose of determining the kinds and amount of mental arithmetic exercises now being included in textbooks. Every page, though in some cases it was only a part of the page, on which mental arithmetic exercises occurred was counted. Pages designated for oral class discussion for purposes of developing understanding of a new precedure, to share thinking, or to review a procedure were not included.

It can be noted that each of the six books included some computation practice in estimating answers, problem-sloving practice in estimating answers, and reading and using graphs and scales. Only one textbook included exercises on interpreting quantitative statements using "reference measures." Systematic practice in rounding numbers was not generally given in these fifth grade texts except as this was a necessary step in the practice in estimating answers.

Each of the books ranged in number of pages from about 300 to 350. The highest number of pages in a single book providing mental arithmetic exercises as used in this article was found to be forty-three, with the lowest being nine pages. From book to book there was found to be considerable variation as to the number of pages presenting exercises of each type listed in Table I. Only one of the books provided exercises with all of the types of exercises listed in the table.

Conclusions and Comments

1. Each textbook could be improved in one or several ways regarding the amount and types of mental arithmetic exercises included.

2. All of the textbooks need to give considerably more attention to developing ability to interpret the quantitative statements occurring in reading reference materials.

3. Children need to be helped to realize the advantages in rounding numbers and they need experience in judging when to use rounded numbers and when to use exact numbers.

Table I

Mental Arithmetic Exercises in Six Fifth-Grade Textbooks

Type of Exercise	Number of Pages of Each Type					
	Textbook A	Textbook B	Textbook C	Textbook D	Textbook E	Textbook F
Computation practice (without paper and pencil, exact answers)	2	7	13	2	0	6
Problem-solving practice (without paper and pencil, exact answers)	0	6	2	1	0	3
Practice in rounding numbers	0	1	0	0	0	1
Computation practice (estimated answers)	2	9	1	5	1	1
Problem-solving practice (estimated answers)	1	5	5	9	1	1
Interpreting quantitative statements, using "reference measures"	0	2	0	0	0	0
Reading and using:						
Tables	0	2	0	2	0	2
Graphs	6	9	4	6	5	5
Scales	6	2	2	6	2	4
Total	17	43	27	31	9	23

4. Many problems encountered in life situations are not in written form. Children need more opportunities to listen to problem situations read or told by others and to solve without the aid of paper and pencil.

5. While several of the textbooks provided without-paper-and-pencil practice in computing for exact answers only two textbooks gave attention to teaching children short-cut ways of handling numbers and performing processes in without-paper-and-pencil situations. For example in performing this division, 5¢/$1.40, without paper and pencil, one short-cut way is to think "how many 5's in 100" and "how many 5's in 40," thus quickly arriving at the answer, 28. Children must be directed in this learning just as in learning the longer ways of performing processes.

6. More testing of children's ability to estimate answers should be included in textbooks. Too often the tests are for exact answers only.

7. A careful analysis of mental arithmetic situations met in life is needed to point up more strongly the need for attention to mental arithmetic in the total arithmetic program and to reveal the kinds of mental arithmetic exercises which should be included in textbooks.

8. A carefully planned mental arithmetic program including varied types of experiences and an evaluation of such a program when put to practice should prove to be a contribution to research in the field of elementary school arithmetic.

EDITOR'S NOTE. Miss Flournoy calls to our attention the need for practice in responding to oral and visual situations rather than responding only to a printed statement. Life is not all written problems, in fact the real value of the written problem is more in the area of the organization of thinking so that one may solve problems regardless of the avenue through which they may be presented. We know that the child tends to learn and to remember those things which he is taught and in which he has practice and experience. Since investigation shows the tremendous role of oral-mental arithmetic in the lives of people, let us provide instruction and practice in this area. Let us recall also that the mental-oral procedure is frequently different from the written procedure and that each child may have a slightly different procedure. We must respect this difference of approach and perhaps even encourage flexibility. The more able pupil usually will have a more direct path to his solution: he may be genuinely inquisitive and seek different ways of solving a problem. Miss Flournoy also points out that mental arithmetic should not be restricted to computations. Likewise, its function should not be one of "disciplining the mind" as was formerly conceived in the older "faculty psychology."

Gibb and Urbancek Two New Associate Editors

Beginning with the November issue of THE ARITHMETIC TEACHER Dr. E. Glenddine Gibb of State Teachers College, Cedar Falls, Iowa will be responsible for book reviews. All commercial materials for review should be sent to her. She will also receive manuscripts of articles from people in her area and will be available for a limited amount of counsel on the development of manuscripts.

Dr. Joseph J. Urbancek of Chicago Teachers College, 6800 Stewart Avenue, Chicago 21, Illinois becomes another associate editor who will have charge of a section of materials which will keep us informed on newer trends and developments in the schools of the country. New courses of study, new experimental programs, new methods of treating certain topics and other developments that teachers and schools may wish to share with their colleagues should be sent to Dr. Urbancek at the address above.

Mrs. Marguerite Brydegaard of San Diego State College, San Diego, California and Dr. John R. Clark of New Hope, Pennsylvania continue as associate editors and will work on the location and development of manuscripts for publication. Manuscripts may be sent directly to them or to the editor.

Non-Occupational Uses of Mathematics

Mental and Written—Approximate and Exact

EDWIN WANDT AND GERALD W. BROWN
Los Angeles State College

A WIDELY ACCEPTED PRINCIPLE of curriculum construction states that school curricula should include experiences similar to, and in preparation for, the problems encountered in adult life. In this regard two aspects of social usage of mathematics have received relatively little attention in curriculum construction or in educational research. These aspects are: (1) the role of mental mathematics, and (2) the role of approximate solutions to problems.

Purpose of the Study

This study was devised for the purpose of obtaining evidence on (1) the relative importance of "mental" as compared with "paper and pencil" mathematics ,and (2) the relative importance of "approximate" as compared with "exact" solutions to problems encountered in everyday non-occupational usage by adults.

Design of the Study

During the 1955–56 school year, the authors requested students enrolled in their classes to keep track of their uses of mathematics for a twenty-four hour period and to report such uses on the special form devised for that purpose. (This form is reproduced at the end of this article.)

The students were instructed to record each mathematical use as it occurred or as soon after as possible. The original design of the study required the students to keep track of their mathematical uses for a period of one week. However, after a pilot group of fifteen subjects attempted to record their mathematical uses for this period, the authors concluded that the task was too demanding for the average subject to do conscientiously for more than a single day. In the interests of assuring more valid basic data, the design of the study was changed to require only a sample of one day's mathematical uses from each subject.

Participation in the study was voluntary. About one-half of the students invited to participate completed the forms and returned them to the authors. Some of the subjects volunteered to have friends, relatives, or acquaintances fill out one of the forms. Although most of the subjects were taking at least one class in college, the population was not made up entirely of college students. Each student was assigned a specific day in the week on which to record his uses, thus insuring approximately equal coverage of the separate days of the week.

A total of 154 subjects returned completed forms. Of these, seven were eliminated for failure to follow instructions, leaving a total of 147 forms which form the basis for the analysis which follows.

Findings of the Study

Of the 147 subjects whose forms were included in the analysis, seventy-five were female and seventy-two were male. Although no attempt was made at a fine break-down in the terms of occupation, the sample was considered to consist of three main occupational groups. One group consisted of sixty-six teachers. A second group consisted of forty-four subjects who indicated no occupation other than that of student or housewife. The remaining thirty-seven subjects were grouped together in an "assorted occupations" group which included such widely different occupations as truck driver, registered nurse, barber, policeman, and IBM operator. Of the 147 subjects, seven were less than twenty years

of age, fifty-eight were in their twenties, fifty-two were in their thirties, and thirty were forty years or older. In general, the group could be characterized as being mature and of better than average education.

The first step in analyzing the data consisted of editing each subject's report to insure that directions had been followed. Since the directions called for mathematical uses *not* directly connected with on-the-job activities, those uses were eliminated which were obviously occupational in nature. Some subjects reported such activities as "telling time." These activities were not considered applications of mathematics for the purpose of this study unless some computation was made utilizing the obtained information.

After the forms were edited, a tabulation was made of the number of mathematical uses reported in each of the following four categories:

1. *Mental-exact.* These uses included only those which were done mentally but required an exact answer.
2. *Mental-approximate.* This category was used for all problems where the computation was performed mentally, and only an approximate answer was necessary.
3. *Paper-and-pencil-exact.* This category was similar to the first one, except the subject used pencil and paper to perform the computation.
4. *Paper-and-pencil-approximate.* These uses included only those where pencil and paper were used by the respondents; however, they did not require exact answers

An example of each type can be found on the form at the end of this report.

The categorizations of the uses were made by the subjects as an integral part of recording their mathematical uses.

A summary of the number of mathematical uses reported in each of the four categories is presented in Table 1.

Table 2 presents the per cent of uses in each of the four categories, mental-exact, mental-approximate, paper-and-pencil-exact, and paper-and-pencil-approximate.

In all, the 147 subjects reported a total of 634 instances of using mathematics. Since each subject recorded his or her uses during one twenty-four hour period, this means that the average person in the group reported somewhat over four uses for the twenty-four hour period. Since there were undoubtedly some applications which were inadvertently omitted in the reports of the subjects, this figure should be considered to be a conservative estimate of the average number of nonoccupational applications of mathematics made daily by the subjects in our sample.

Although it was not the primary purpose of this study to compare the mathematics used by various occupational groups, or by men and women, the data were tabulated separately for each of these groups so that some idea about the generality of the results reported here might be obtained. From inspection of these limited data, there does seem to be a marked similarity between the per cent of usages reported by the various groups in each of the four categories: mental-exact, mental-approximate, paper-and-pencil-exact, and paper-and-pencil-approximate. Though not reported here in detail, an attempt was made to see if there were any obvious differences in the types of uses reported by the various age groups represented in our sample. No correlation with age was apparent in an inspection of the data. Thus, the evidence so far suggests the hypothesis that the per cent of mental mathematics and approximate mathematics used is unrelated to the variables: age, occupation, and sex.

The primary purposes of this study were to determine the relative importance of "mental" and "paper-and-pencil" mathematics, and the relative importance of "exact" and "approximate" mathematics in the solutions of problems encountered in everyday nonoccupational usage by adults. Since 75 per cent of the uses reported were "mental" and 25 per cent "paper-and-pencil," in this study "mental" uses outnumbered "paper-and-pencil" uses in the ratio of 3 to 1. "Approximate" uses constituted 31 per cent, or almost one-third, of all uses reported. It must be noted that almost all of these 31 per cent were "mental-approximate," and that apparently "paper-and-pencil-approximate" uses constitute only a very small percentage of non-occupational mathematical uses.

Table 1
Number of Uses of Mathematics Reported by the 147 Subjects

Occupational Group	Sex	Number	Type of Usage: Mental Exact	Mental Approx.	Paper and Pencil Exact	Paper and Pencil Approx.	Total
Teachers	F	27	66	29	27	1	124
	M	39	65	43	34	8	150
Students and Housewives	F	31	70	36	30	4	140
	M	13	34	15	9	4	62
Assorted Occupations	F	17	29	19	18	0	66
	M	20	43	27	16	6	92
All Occupations	F	75	165	84	75	6	330
	M	72	142	85	59	18	304
Total Group		147	307	169	134	24	634

Table 2
Per Cent of Each Type of Usage Reported by 147 Subjects

Occupational Group	Sex	Number	Type of Usage: Mental Exact	Mental Approx.	Paper and Pencil Exact	Paper and Pencil Approx.	Total
Teachers	F	27	53%	23%	22%	2%	100%
	M	39	43%	29%	23%	5%	100%
Students and Housewives	F	31	50%	26%	21%	3%	100%
	M	13	55%	24%	15%	6%	100%
Assorted Occupations	F	17	44%	29%	27%	0%	100%
	M	20	47%	29%	17%	7%	100%
All Occupations	F	75	50%	25%	23%	2%	100%
	M	72	47%	28%	19%	6%	100%
Total Group		147	48%	27%	21%	4%	100%

Implications of the Study

Although the sample employed in this study is by no means representative of our national population, there appears to be a good probability that mental applications of mathematics do outnumber paper and pencil applications in the general population, and that approximations do enter into a sizeable percentage of everyday applications of mathematics.

If the findings of this study are reinforced by similar findings with other groups, the implication will be clear that considerable emphasis should be placed on mental and approximate mathematics at both elementary and secondary levels. This emphasis would include not only practice in the mental arithmetic process, but also applications to life situations.

Readers are invited to make use of the form used in this project if interested in studying the problem further. Results of additional research in this area would be of great interest to the authors.

Survey of Mathematical Usage*

The following four examples illustrate the method of recording your uses of mathematics. Notice that the solutions to the problems are not called for, only the problems themselves.

Examples (check one column for each problem)

The problem	Mental		Paper and Pencil	
	Exact	Approx.	Exact	Approx.
Bought Sunday paper for 20¢, gave boy $1.00, calculated the change	X			
Wrote check for $6.26—previous balance $326.24—calculated new balance			X	
Figured approximate cost to carpet living room 18′6″×14′ with carpeting costing $6.85 per sq. yd.				X
Compared the cost of 1 lb. 6 oz. of soap powder at 30¢ with cost of 3 lb. box of soap powder at 59¢		X		

* Only a portion of the survey form is printed here. This shows the essential nature of the categories used. Directions are available from the authors.

EDITOR'S NOTE. Professors Wandt and Brown set out to make a more comprehensive survey which would have taken the activity of each individual through one week but found this to be too much of a task for the participants and so settled for a one-day period but arranged that the one day should be a different day of the week for subgroups of their sample. Their sample is much more representative than one would expect from the way in which it was chosen. The editor wishes they had followed the original plan. He wishes also that some very competent person who would be invisible to each participant might have been ever present to note the opportunities for use of mathematics which the participant in the study failed to note and even surmise. Perhaps others will accept the invitation of the authors to collect and report more data in a similar study.

The authors restricted their cases to applications of mathematics in which some type of computation was involved. This is a limiting factor which they chose deliberately. It is suggested that other investigators include such uses of mathematics as recognition and thinking with simple concepts and principles, e.g., telling time, reading a gauge, counting money, days, etc., determining a future date, weighing and measuring, etc., etc. This is not a simple and easy study but one that is tremendously worthwhile especially if *opportunities for use* as well as *actual uses* are recorded and if the participant's attitude and action are also reported. Who will add to the data of Messrs. Wandt and Brown?

The implications for schools are apparent in Tables 1 and 2. Let us provide experience and practice in "mental" as well as in paper-and-pencil arithmetic.

THE LANGUAGE OF DIVISION

Division is usually indicated in one of two ways: $3\overline{)6}$ or $6\div3$. We divide by the initial number when using the frame and by the second number when we use the division symbol. This causes some pupil confusion in deciding which number to use as the "operator" and this is particularly disturbing in the division of fractions when one fraction must be "inverted." The teacher must stress the meaning of arithmetic expressions in written form as for example $3\overline{)6}$ as "3 divides 6" and $6\div3$ as 6 divided by 3. In English class pupils learn of the active and passive cases.

John Rides the bicycle (active) 3 divides 6
The bicycle is ridden by John (passive) 6 divided by 3

If the subject is passive, it does nothing. We must decide which number is doing the work. Similarly with addition, subtraction, and multiplication, it is worthwhile analyzing the meaning of statements with mathematical symbols. This will save later confusions.

Contributed by WILBUR HIBBARD
Highland Park, New Jersey

The Number System and the Teacher

ANN C. PETERS
State Teachers College, Keene, New Hampshire

THE FOLLOWING NEWS ITEM appeared in a recent issue of one of the nation's leading newspapers:[1]

> **FEEL FOR SCIENCE DEVELOPS IN YOUTH**
>
> LOS ANGELES (UP)—The future of many American scientists depends on how they feel about mathematics in the fifth and seventh grades, according to a professor at the University of California at Los Angeles.
>
> He polled 459 selected junior high school pupils who indicated that the fifth and seventh grades were the crucial testing areas in their liking or disliking mathematics.
>
> He said that the pupils' attitude toward the basic science solidified during these years, although the third and eighth grades also were important areas of influence.
>
> Pupils indicated they liked mathematics because they realized it was important in future life and because they enjoyed working out problems.

Undoubtedly, how one feels about mathematics in his early school years determines to a great extent its immediate and future usefulness. As educators, however, we might well question the nature of "problems" and begin to realize that the overly practical approach may be a dangerously limiting one and ultimately our own undoing. Some teachers maintain there are both the conceptual and the practical operations inherent in thinking and that both of these are to be found in problems of various kinds. When education on all levels in these United States faces the theoretical as well as the practical, the children of better than average ability may "feel" even better than they now do about mathematics. Perhaps we have placed too much emphasis upon the manipution of symbols and application of mathematics even in the "meaningful" approach to numbers. The practical aspect of mathematics is essential for the mundane affairs of day-to-day living and these numbers *do* represent some physical quantity. Moreover, this is exactly what is *not* done in modern mathematics—the kind of problems mathematicians are working on today. How to bridge the gap between traditional mathematics as most of us know it and modern mathematics as contemporary mathematicians know it seems to be one of the dilemmas of our educational age.

Elementary school mathematics has a body of knowledge which is good to know for its own sake. Teachers, however, are seemingly reluctant to condone mathematics for mathematics sake. Yet no one frowns upon life for life's sake! Dantzig holds that "our school curricula by stripping mathematics of its cultural content and leaving a bare skeleton of technicalities have repelled many a fine mind."[2] Truly, mathematics may always be one of the last frontiers because of its challenge to the human thinking. It is a "constantly expanding subject in which *ideas* and not manipulations play the dominant role."[3] But, our children need *time* to think and they need mathematical substance to think about. (Teachers, too, plead for time to study and time to teach!) Self-discovery, inventiveness and creativity are all permissible and highly desirable when children at all grade levels are exposed to our decimal number system. Even to wonder about the origin of number and about the possible condition of man today had he invented some other system stimulates speculation and imagination. Could it be our nation today is lacking in theoretical mathematicians because most of our students at all

[1] New York *Times*, Monday, February 18, 1957.

[2] Tobias Dantzig, *Number, the Language of Science*. New York, The Macmillan Company, 1930.

[3] Roy Dubisch, *The Nature of Number*. New York, The Ronald Press, 1952.

levels of instruction have little or no sense of direction in this field even after they have been exposed to the traditional curriculum for ten or more years?

If number is a system of ideas, then it behooves teachers to consciously guide children *to* the system and *to* the ideas inherent within the structure and framework of the system. Meaningful mathematics is submerged within the structure of number and awaits the discovery of each human mind to its combinations, collections, operations and laws, and eventually to the symbols that express these ideas.

The Collectional Approach

Man originally discovered number in answer to two ever-prevailing questions: "How many?" and, "How much?" To appreciate the difficulty of representing quantity is no easy task for modern man.[4] It seems evident, however, that early man created his number systems in the image of his fingers and/or toes. Anthropologists have found number "bases" of five, ten, or twenty in widely separated cultures with base ten the most common and base five still prevalent (Greenland Eskimos). But even before a base was established, centuries of time were consumed between the discovery for twoness and fiveness, and fiveness and tenness. The collection six, the idea six, and the symbol six represent inventions of great ingenuity. More significant still was man's collecting ten (possibly sticks) into one bundle, giving him the alternative of ten ones or one ten; thus establishing base ten as the civilized world has come to accept it. The eventual pebbles (calculus) laid in sand grooves and the invention of the abacus with its 1/10 ratio in a positional value system testify to the modern child's need to relive a similar experience and to verify for himself the essential nature of number. The privilege should be his to see and to feel that in terms of "sticks" or "stones" each provides three different ways to show one hundred! The cardinality of number is a matching process rather than counting, be the idea six, ten or a hundred.

To free himself of collections of "things" man invented symbols to *represent* positional quantity of things and, thus, his *ideas* about number became abstract. In his base ten system, the symbols 5, 7, and 2 could be utilized to represent six distinctly different positive integers: 275, 257, 527, 572, 725 and 752. Any number system is, of necessity, a neat and compact system having a structure and ratio not always obvious to the user. Perhaps base 10 can take on additional meaning when viewed in table form, keeping in mind that *one* (or ones) is always the essential number (regardless of the size of the base):

[4] Louis Karpinski. *The History of Arithmetic.* New York, Rand, 1925.

TABLE I

1,000 ones = 1 thousand
100 ones = 1 hundred
10 ones = 1 ten
1 one
.1 = 1 tenth
.01 = 1 hundredth
.001 = 1 thousandth
10:1 ratio

TABLE II

1 thousand
10 hundred
100 tens
1,000 ones
10,000 tenths
100,000 hundredths
1,000,000 thousandths
All are equivalent

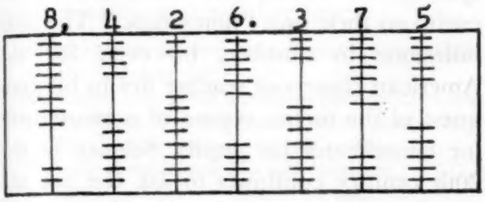
TABLE III

675 = .675 thousands
675 = 6.75 hundreds
675 = 67.5 tens
675 = 675 ones
675 = 6,750 tenths
675 = 67,500 hundredths
All are equivalent

TABLE IV

.0675 thousands
.675 hundreds
6.75 tens
67.5 ones
675 tenths
6750 hundredths
All are equivalent

Another implication in number can be noted when comparing the relationship between quantity expressed on the abacus and again in symbols.



Since our system is structured on ten and powers of ten dependent on positional value, we admit the following quantity on order: 8 thousands and 4 hundreds and 2 tens and 6 *ones* and 3 tenths and 7 hundredths and 5 thousandths, or stated in another way:

$$8(10^3)+4(10^2)+2(10^1)+6(10^0)+3(10^{-1})$$
$$+7(10^{-2})+5(10^{-3}).$$

Evident in number is design, symmetry, balance, and, above all, a structure enabling man to represent quantity, thanks to the invention of zero, no matter how infinitely large or infinitesimally small. And note, the common fraction $3/8 = .375$, a rational number, and may be expressed as negative powers of ten: $3(10^{-1})+7(10^{-2})+5(10^{-3})$.

Civilized people everywhere today use base ten and if Providence had endowed us with eight or twelve fingers (digits) we undoubtedly would have had something other than the decimal system. Base ten, while adequate, is not ideal. Base 12 seems the ideal in that twelve has twice as many factors (2, 3, 4 and 6) as ten (2 and 5) thus providing greater flexibility and convenience within the system and especially in working with fractions. Two additional symbols "t" and "e" must be employed in base 12 as shown in the following table:

Decimal	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Duodecimal	1	2	3	4	5	6	7	8	9	t	e	11	11	12	13

10

The "bundle" of 12 is a dozen and a dozen and a half is 16!

Teachers themselves need to be familiar with other number bases. Better still, if they can operate with ease in another system, say the quinary (base five), teachers might well appreciate the challenge the child meets in his elementary school mathematics. If the "bundle" has five the scale becomes thus:

Decimal	1	2	3	4	5	6	7	8	9	10
Quinary	1	2	3	4	10	11	12	13	14	20

Seven cents is 1 nickel and 2 pennies! The computation of the civilized world could be accomplished in the quinary system but not as efficiently as in its present decimal base. For instance, our 4,683 becomes 112,213 in base five.

The history of the binary system, base two, from the time of Leibnitz, its discoverer, to the invention of the dial telephone and now the modern electronic computing machine reads like a fascinating novel. The system has but two symbols, 0 and 1, to express all quantity.

Decimal	1	2	3	4	5	6	7	8
Binary	1	10	11	100	101	110	111	1000

Now we see the algorism, $1+1=2$ only in number systems having base three or greater. Base two has advantages—economy of operations and simple tables thus:

$$1+1=10 \quad \text{and} \quad 1\times 1=1$$

The disadvantage is the lack of compactness. The decimal number $64=2^6$ would be expressed in the binary system as 1,000,000. It is said that the mystic elegance of the binary system made Leibnitz (1647–1716) exclaim: "One suffices to derive all out of nothing!" In his binary arithmetic he saw the image of Creation. He imagined that Unity represented God, and Zero the void and that the Supreme Being drew all beings from the void, just as unity and zero express all numbers in his system of numeration. Were Leibnitz to behold the modern electronic computing machines, the Univac, today! Moreover, we must admit now that $1+1=2$ is true only in number systems having base three or greater.

Our decimal number system has its limitations as previously noted, however, as teachers we can appreciate the structure of *any* system. The idea of "bundle" or base, of positional value or ratio, of design and symmetry, pertains to any system regardless of the size of the base. The same fundamental operations may be performed in each system and the laws of the decimal number system would seem applicable to most other systems. Familiarity with other number systems should provide teachers not only with a greater appreciation of our own decimal system, but should give them greater security on their own "home base." The next milestone in number, however, for the American classroom teacher lies in his conquest of the metric system of measurement for himself and his pupils. Science in the 20th century continues to ask the age old questions: How many? and, How much? Man has invented the decimal number system to answer the former and the decimal measuring system to answer the latter. The understanding and use of the metric system in the junior high school age group in this country would give secondary and collegiate science the proverbial "shot in the arm."

The Geometric Approach

In contrast to collections and powers, and to ratios and positional value, the number system may be described as an unending scale or line stretching to infinity on either side of a starting point called zero. This approach to number admits negative values as well as positive. Elementary arithmetic is usually limited to the operations with positive numbers. *Thinking about number*, however, may utilize both positive and negative values as well as positive and negative directions. Indeed, it is to be encouraged.

The line approach emphasizes the ordinal or serial meaning of number though the cardinality of number is also provided for at each specific point equally spaced and with a named value. It accommodates fractions as well as whole numbers. If the number line has neither a first nor last integer, neither has it a definable count on all the fractions that can be represented between each of the cardinal points. But in spite of the seemingly compact structure of the number line by now, it is "full of gaps." The rational numbers (integers and fractions) still have spaces between each point which we define as nondimensional. These spaces may be filled with another kind of number, the irrational, which, too, are entitled to points in this number line. The irrationals are, indeed, such for down through the ages they have been man's unknown quantities. Among them are $\sqrt{2}$, $\sqrt{3}$, π, e, and others. The rational *and* the irrational make up what is

known as the real number domain. Any real number can be represented by a point on a line and, conversely, any real number can be assigned to any point on a line.[5]

There are certain characteristics, then, relative to the real number line:

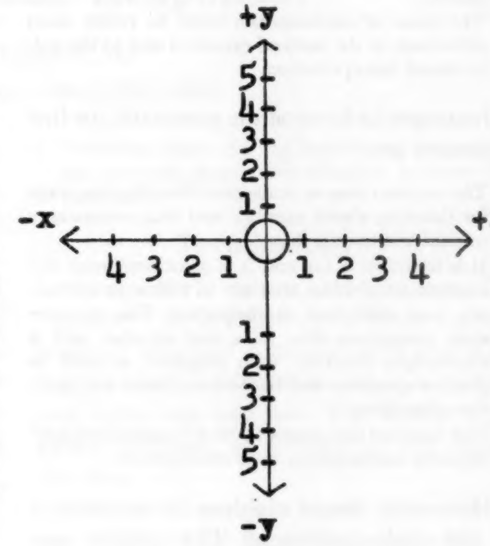
1. *All* real numbers have been arranged (conceived) in their order of size.
2. No matter how great a positive (or negative) real number there is one greater and no matter how small a number there is one still smaller.
3. The real domain has two sub-divisions:
   a. The rational. b. The irrational
4. The total of real numbers is everywhere dense. Between any two real numbers an infinite number of other real numbers may be inserted.

The total of real numbers is not only well ordered and compact, but it is *perfect*. Not every compact aggregate is perfect as the rational domain indicates, but every perfect aggregate is compact, as Cantor proved (1884). This condition he defined as a continuum. The real number domain is the *number continuum* or, in everyday parlance, the number line. Perhaps an outline may further clarify the perfect aggregate.

Cantor's Number Continuum

- I. Real Numbers
  - A. Rational numbers (positive and negative)
    - 1. Integers
    - 2. Fractions
      - a. Common and decimal
        - 1. terminating (.375)
        - 2. repeating $(.\dot{4}\dot{5})$
  - B. Irrational numbers (positive and negative)
    - 1. Elementary (Algebraic: $\sqrt{2}$)
    - 2. Transcendental
      - a. π b. e
      - c. trigonometric ratios
      - d. logarithms
  - C. Zero
- II. Imaginary Numbers

Note, please, zero is a very respectable number in the continuum. The real number continuum we shall now identify as the x-line of the coordinates. Descartes (1640) discovered the coordinates attributing the rational numbers to the x-line and the imaginary numbers, ($\sqrt{-9}$ or $3\sqrt{-1}$ or 3i), to the y-line. This unification of arithmetic and geometry (analytical geometry) was a tremendous influence on the development of mathematical thought. It is the essence of map-making and much else.



What are the implications of the line approach to number for teachers? Turn to the objective world. Whether we use a ruler or a weighing balance, a pressure gauge or a thermometer, a compass or an altimeter, we are always measuring what appears to us to be a continuum, and we are measuring it by means of a graduated *number scale*—the arithmetic of real numbers. The number line over our blackboards and the ruler on the child's desk has many untapped possibilities for the teaching of arithmetic!

Which Approach?

We teachers need to be conscious of both the collectional and the line approach to number. We need to use both approaches and likewise teach children to be sensitive to how they employ number. Admittedly the collectional approach began with the dawn of civilization, it is an aspect of the child's pre-school world and it is his first contact with number in his formal education. There seem to be three advantages favoring the collectional meaning:

[5] Courant, R. and Robbins, H. *What Is Mathematics?* New York, Oxford University Press, 1941.

1. It is easier to think of the basic operations as the grouping and re-grouping of collections and hence, to rationalize the written algorism.
2. Positional notation of number in the decimal system emphasizes collections of tens. This represents an historical heritage which cannot be ignored.
3. The laws of mathematics seem to relate most effectively to the natural numbers and to the collectional interpretation.

Advantages in favor of the geometric, or line approach are:

1. The number line or scale provides effective ways for thinking about number and thus encourages mental arithmetic.
2. It is foundation for and it is associated with the mathematical ideas that are to follow in secondary and collegiate mathematics. The number scale recognizes zero as a real number and it encourages thinking with negative as well as positive quantity and likewise negative and positive directions.
3. The number line seems to be a "natural bridge" between mathematics itself and science.

How soon should children be introduced to the scale approach? The child's own maturity and the teacher's best judgment may need to be the deciding factor. It would seem the average second grader uses the ruler with handiness and intelligence. Indeed, many children discover the number scale for themselves and use it in working out their combinations. The thermometer has by chance aided many youngsters in obtaining a sense of direction in the use of directed numbers. It is assumed, however, that class (guided) instruction increases a child's efficiency in dealing with number from both approaches. Numbers, points, etc. are not substantial things in themselves —they only state the interrelationship between mathematically "undefined objects" and the rules governing operation with them. What points, numbers and lines "actually" *are* cannot and need not be discussed in mathematical science. The secret of the matter is structure and interrelationship, and eventually, laws and limitations. Number has property or characteristics and may involve action. The combination of positive integers results in operations called addition and multiplication. An operation is an *act*.

Creative Teaching and Learning

Guided instruction in number should provide the opportunity to think about number creatively. The various approaches to the solution of a problem in which each child discovers for himself a reasonable conclusion is in a sense creative thinking. Since the late 1920's William H. Kilpatrick and others have maintained that all learning is creative because the child acquires new insight or reorganizes his behavior in a way that is *new* for him. Most educators today accept this broad concept of creativity. But while each child discovers new knowledge, he does not create it.

The other aspect of creativity, original discovery, is the sum-total of all human mathematical understanding. It involves problem solving plus critical thinking and imagination—the recombination of known elements or ideas into something new. Ours is the difficult problem of how to stimulate and guide such creative effort which may begin somewhat earlier than once supposed, but is usually associated with later childhood, adolescence, and maturity. Russell feels (p. 323–24):[6]

> The range of scientific knowledge is so great today that an original discovery growing out of this knowledge before the age of twenty is almost impossible, and most new findings must come some years later than that. The good typical school program, nevertheless, assumes that the individual will still do some creative thinking in such fields as natural science, social science and mathematics in both elementary and secondary schools.

The teaching of number as a system and with a sense of direction may be effectively planned and directed toward experimentation and creative production if the teacher herself sees number as a system of ideas and provides materials, time, and the atmosphere for creative work. This in no way means every bright child should become a mathematician or scientist. But it does mean the opportunities should be provided especially for the gifted, if he is so inclined.

[6] Russell, David H., *Children's Thinking*. Boston, Ginn and Company, 1956.

(*Editor's Note on page 167*)

100% Automatic Response?

Louis E. Ulrich, Sr.
William T. Sherman School, Milwaukee

It was not so long ago that statements such as the following were current: "The only right standard for drill is the 100% standard and this is easily possible in arithmetic because the processes calling for drill are few. The harm done by a lower standard is immeasurable and defeats the chief purpose of drill. The drill load must be small to make the 100% standard possible for every child."[1] This writer, in his capacity as a teacher, administrator, and researcher finds it difficult to agree with the above quotation.

An Experiment in Multiplication

This experiment on the multiplication of a two-figure multiplicand by a two-figure multiplier was undertaken in grade 5B to verify previous experience and also to help teachers form their own conclusions by giving them some objective data. The questions sought to be answered within the limits of this experiment were:

1. What are the arithmetic processes which the decimal system of notation and computation requires the student to learn in this area of multiplication?
2. Are the processes few and discrete or do they form numerous patterns that show relationships?
3. After having learned to multiply this area meaningfully,[2] for how long will the class still make errors, and if they do, will the students be able to discover and correct such errors without help from the teacher?
4. Is it reasonably possible with the teacher's help to get 100% automatic response for every pupil in the class?
5. Will children miss an example that was previously done correctly?
6. Were the errors due to inability to understand the processes, or are other elements involved that cause error?

Terminology

"Frequency" means how many. The "number" 25 is made up of seven decimal units—2 tens and 5 ones. The 2 and 5 are "frequencies" of the different valued decimal units—*one* and *tens*. The words: digit, figure, numeral, or number do not express this idea.

In order to conserve space, abbreviations are employed for the following words:

"Transform"—T—means to change the notation but not the value. Thus, 1111111111 is transformed to one decimal unit and written as 10, meaning one ten and no ones. In subtraction and division, transformation is in the reverse. "Carry" and "borrow" do not express this idea,[3] but refer to what was done on the abacus where counters were used.

"Remember"—R—means to remember the unseen result of transformation so that it can be added to the seen or unseen frequencies of the next column.

"Add"—A—means to add a seen or unseen frequency to a seen or unseen frequency.

"Dot"—.—means to write a frequency in the answer.

"Multiply"—M

"Bring down"—b

This area of multiplication contains 8100 examples divided into four apparent structure groups as follows:[4]

Illustration A

I. (*941 cases*)

```
   24
  ×12
  ---
   48   M.M.
  24    M.M.
  ---
  288   b.A.b.
```

Type 1 illustrating Group I.[4]

II. (*1018 cases*)

```
   21
  ×46
  ---
  126   M.M.T.
  84    M.M.
  ---
  966   b.A.A.
```

Type 12 illustrating Group II.

III. (*1632 cases*)

```
    39
   ×52
   ---
    78   MTR.MA.
  195    MTR.MAT.
  ----
  2028   b.ATR.ATRA.
```

Type 32 illustrating Group III.

IV. (*4509 cases*)

```
    99
   ×72
   ---
   198   MTR.MAT.
  693    MTR.MAT.
  ----
  7128   b.ATR.ARATR.A.
```

Type 48 illustrating Group IV.

Some reaction to drill and analysis is current and we find writers who oppose analysis of algorisms into their respective processes because, they say, analysis divides computation into innumerable unrelated parts which are then fixed by meaningless mechanical drill. Analysis is also said to dissect without putting the whole together again.[5] This position is sometimes taken in spite of the fact that meanings in arithmetic computation are determined by the Decimal System of Computation.[6] The examples in Illustration A show us that while the individual processes are indeed few they do unite to form process patterns and that these patterns increase in complexity to form an example complexity scale. See Illustration B. Analysis therefore when thoroughly done consistent with the decimal system shows relationships, and enables classifications of examples requiring identical patterns into types so that the teacher's efforts may be concentrated upon a few recognized type patterns rather than flounder among a heterogenous mass of seemingly unrelated examples.

It should be stated here that synthesis without analysis is impossible. In terms of education, this means that teaching, diagnosis, and remedial instruction, as well as practice is not readily accomplished when relationships as disclosed by process and their patterns are not understood.

Each of these four structures (see Illustration A) again are divided into a specific number of types each made up of a definite number of examples (cases). All the examples of a particular type (see Illustrations B and C) require the same process pattern for their computation and notation so that there should be little difference in the required ability to work any example of a particular type. The teacher of this class* taught all preceeding types meaningfully in order of their complexity so as to take advantage of transfer of learning.[7]

* Miss Elvera Ankerson, Wm. T. Sherman School, Milwaukee, Wisconsin

The class was then tested on 15 different days on ten examples of the last or 48th type which requires a combination of the most complex process patterns. A test on this type would show the level of achievement since the process patterns of this type really include the processes of all preceeding types (see Illustrations B and C) There was no time limit set.

The first ten tests were given without any help from the teacher, and the students did not see the test results. The last five tests were each given after help from the teacher on the preceeding test. The only variables present were the changes in the position of each example in each succeeding test, and in the change in the numbers of each of the ten examples. The process patterns for all the examples were identical and fixed, since they are of the same type.

The four basic patterns below combine to make up all other multiplication patterns of this area.

Illustration B
Multiplication Scale

1. (*23 cases*)

$$\begin{array}{rl} 3 & \\ \times 2 & \\ \hline 6 & \text{M.} \end{array}$$

basic

2. (*58 cases*)

$$\begin{array}{rl} 6 & \\ \times 3 & \\ \hline 18 & \text{M.T.} \end{array}$$

basic

3. (*138 cases*)

$$\begin{array}{rl} 24 & \\ \times 2 & \\ \hline 48 & \text{M.M.} \end{array}$$

combination pattern

4. (*150 cases*)

$$\begin{array}{rl} 43 & \\ \times 3 & \\ \hline 129 & \text{M.M.T.} \end{array}$$

combination pattern

5. (*58 cases*)

$$\begin{array}{rl} 27 & \\ \times 2 & \\ \hline 54 & \text{MTR.MA.} \end{array}$$

basic pattern

6. (*464 cases*)

$$\begin{array}{rl} 37 & \\ \times 3 & \\ \hline 111 & \text{MTR.MAT.} \end{array}$$

basic pattern

Notice how these processes are all combined in the process patterns of the last or 48th type as in Illustration C.

Illustration C
Type 48 analyzed.

There are 719 cases subsumed by type 48.

$$\begin{array}{rl} 66 & \\ \times 69 & \\ \hline 594 & \text{MTR.MAT.} \\ 396 & \text{MTR.MAT.} \\ \hline 4554 & \text{b.ATR.ARARTR.A} \end{array}$$

Notice that there are twenty-two individual processes or twenty-two possible errors, excluding those possible in writing the numbers. One would not say that "the processes calling for drill are few."

A sample of the test follows. Notice that there can be no zeros in the multiplication patterns of this type since a zero means the absence of frequency. Without frequency there can be no transformation.[8]

Illustration D
Test

(1) $\begin{array}{r} 96 \\ \times 22 \\ \hline \end{array}$	(2) $\begin{array}{r} 59 \\ \times 34 \\ \hline \end{array}$	(3) $\begin{array}{r} 65 \\ \times 39 \\ \hline \end{array}$	(4) $\begin{array}{r} 84 \\ \times 24 \\ \hline \end{array}$
(5) $\begin{array}{r} 44 \\ \times 48 \\ \hline \end{array}$	(6) $\begin{array}{r} 58 \\ \times 69 \\ \hline \end{array}$	(7) $\begin{array}{r} 57 \\ \times 72 \\ \hline \end{array}$	(8) $\begin{array}{r} 47 \\ \times 48 \\ \hline \end{array}$
(9) $\begin{array}{r} 95 \\ \times 53 \\ \hline \end{array}$	(10) $\begin{array}{r} 62 \\ \times 98 \\ \hline \end{array}$		

The Tabulation E gives the results on the fifteen tests, each test considered as a unit. The errors on individual examples are not shown here but are discussed later.

The greatest number of tests correctly done (88%) was reached in the 12th try (see Tabulation E). The 8th test was better than the 9th, 11th, and 13th. Additional teacher help in the last five tests did not succeed in getting a perfect class score.

Tabulation E

			Test Number																	
A	B	C	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	D	E	F
1	126	150	(No. of Ex. missed on each test)															15	—	—
2	121	142		1	1		1	1	1		1	1			1			7	13	8
3	120	136	4	3		1					1		3		2			9	13	14
4	119	150																15	—	—
5	119	144		2		1	1	1			1							10	9	6
6	117	143	2	1		1		2	1									10	8	7
7	117	148	1				1											13	6	2
8	116	148		1			1											13	6	2
9	116	147		1				1					1					12	12	3
10	116	137	3	1	1	2	3	1		1	1							8	9	13
11	115	144			1		3			1			1					11	11	6
12	115	142	2				2	4										12	7	8
13	115	142			1	1	1		1	1	1	1				1		7	14	8
14	113	148						2										14	6	2
15	112	133	2	2	2	2	3	1		1		1			2		1	5	15	17
16	112	141	2		1	2	3	1										9	7	9
17	111	148	1		1													13	3	2
18	110	145	1			1	1	1									1	10	15	5
19	108	131	2		5	1	1			3		1		1		1	4	6	15	19
20	107	142	1				1	2		1			1			2		9	14	8
21	106	104	2	4	6	6	7	3	3		6	4	2	1	2			3	13	46
22	105	138	2	1		1	1	2	1		1		1	1		1		5	14	12
23	104	137	2	1	2	1			1				4		1		1	7	15	13
24	103	149												1				14	12	1
25	103	147	1					1	1									12	7	3
26	102	139	3	5	1			2										11	7	11
27	101	145	2		1	1							1					11	11	5
28	100	147	1		1		1											12	5	3
29	99	135	1	3	1	1		3	2		2		1		1			6	13	15
30	98	135	3	2	5	1	1						1		2			8	13	15
31	95	139	3	3		3											2	11	15	11
32	95	136	1	2	2	1	1	1	2	1		2			1			5	13	14
	G	4512	10	16	16	15	14	16	21	25	24	26	22	28	24	28	27	313	321	288
	H	%	31	50	50	47	44	50	66	78	75	81	69	88	75	88	84			
			Number and per cent of each test correct																	

A. Pupil class number
B. Intelligence quotient
C. Number of examples correct
D. Number of perfect scores
E. Last perfect score
F. Number of examples incorrect
G. Total
H. % of pupils who got the test correct

$\frac{4512}{4800} = 94\%$ correct

$\frac{288}{4800} = 6\%$ incorrect

*Tabulation F**

Notice that 2 children got 15 tests correct.
2 children got 14 tests correct.
3 children got 13 tests correct.
4 children got 12 tests correct.
4 children got 11 tests correct.
Mean.....3 children got 10 tests correct.
3 children got 9 tests correct.
2 children got 8 tests correct.
3 children got 7 tests correct.
2 children got 6 tests correct.
3 children got 5 tests correct.
1 child got 3 tests correct.

* Based on Tabulation E.

Tabulation E also shows:

1. That every child got all the examples of one or more tests correct at sometime.
2. All children but two made errors after they had previously gotten all the examples of at least one test correct.
3. By the time the children finished the 10th test (see Tabulation E) 26 out of 32 children or 81% had received a perfect score. This shows that children did very well in discovering and correcting their own errors.
4. Some children with an I.Q. below 100 did better than some above 100.
5. The I.Q. shows only a general class tendency but does not always prognosticate as to the possible achievements of any particular child.
6. The children evidently knew how to work all the examples hence the errors were due to other causes than inability to work the examples.

A study of the completed tests sheets shows that examples were missed as follows:

Illustration G

Example 1 missed	7 times out of	150 times.
Example 2 missed	32 times out of	150 times.
Example 3 missed	48 times out of	150 times.
Example 4 missed	18 times out of	150 times.
Example 5 missed	18 times out of	150 times.
Example 6 missed	27 times out of	150 times.
Example 7 missed	22 times out of	150 times.
Example 8 missed	51 times out of	150 times.
Example 9 missed	22 times out of	150 times.
Example 10 missed	43 times out of	150 times.
	288	4800

Out of 4800 examples, 288 or 6% were missed by 32 children. And 94% correct is quite satisfactory.

In studying the individual errors made, especially in examples 2, 3, 8 and 10, it was found that with few exceptions they occurred when children failed to add the transformed frequency either in the multiplication or addition patterns. Illustrations of this error are seen in the following examples:

$$\begin{array}{r} 62 \\ \times 98 \\ \hline \rightarrow 436 \\ 558 \\ \hline 6016 \end{array}$$

Here the "remember" caused an error. 48+1=43, a fact which the child knew how to recapture by counting.[9] It appears that this child relied on automatic response unchecked by reason.

$$\begin{array}{r} 95 \\ \times 53 \\ \hline \rightarrow 295 \\ 475 \\ \hline 5045 \end{array}$$

Here the error was 3×9=27+1=29.

$$\begin{array}{r} 58 \\ \times 69 \\ \hline 522 \\ 348 \\ \hline \rightarrow 3002 \end{array}$$

Here the error was 3+1=3.

$$\begin{array}{r} 62 \\ \times 98 \\ \hline 496 \\ 358 \\ \hline \rightarrow 6176 \end{array}$$

Here the error was 1+4+5=11.

$$\begin{array}{r} 65 \\ \times 39 \\ \hline 585 \\ \rightarrow 185 \\ \hline 2435 \end{array}$$

Here the error was 3×6=18+1=18.

$$\begin{array}{r} 57 \\ \times 72 \\ \hline 114 \\ 399 \\ \hline \rightarrow 3104 \end{array}$$

Here the error was 1+3=3.

$$\begin{array}{r} 59 \\ \times 34 \\ \hline 236 \\ 178\leftarrow \\ \hline 2016 \end{array}$$

For some reason or other 3×9=18 was an error made frequently.

$$\begin{array}{r} 59 \\ \times 34 \\ \hline 236 \\ 168\leftarrow \\ \hline 1916 \end{array} \qquad \begin{array}{r} 95 \\ \times 53 \\ \hline \rightarrow 195 \\ 475 \\ \hline 4945 \end{array}$$

Once in a while children made a mistake in "bringing down," as:

$$\begin{array}{r} 65 \\ \times 39 \\ \hline 585 \\ 195 \\ \hline 2536\leftarrow \end{array}$$

Sometimes an error is repeated in succeeding examples when conditions are somewhat similar, as:

$$\begin{array}{r} 88 \\ \times 24 \\ \hline \rightarrow 342 \\ 178 \\ \hline 2102 \end{array} \qquad \begin{array}{r} 44 \\ \times 48 \\ \hline \rightarrow 342 \\ 176 \\ \hline 2102 \end{array}$$

Sometimes children are influenced to write the frequency they see elsewhere as the 6 here:

$$\begin{array}{r} 96 \\ \times 22 \\ \hline 192 \\ 196\leftarrow \\ \hline 2152 \end{array} \qquad \begin{array}{r} 96 \\ \times 22 \\ \hline 196 \\ 196\leftarrow \\ \hline 2156 \end{array}$$

The figure 9 seems to be a difficult frequency to which to add a remembered frequency as:

$$\begin{array}{r} 65 \\ \times 39 \\ \hline 585 \\ 195 \\ \hline 2635 \\ \nearrow \end{array}$$

Teacher's Findings in Her Questioning During Special Help With These Children

Pupil No. 2. Does not work up to capacity in any subject.
Pupil No. 6. I.Q. is too high for him.
Pupil No. 10. Very careless.
Pupil No. 12. A spotty worker—should have made a perfect score.
Pupil No. 13. Missed same example in four tests.
Pupil No. 14. Should have made a perfect score but made same error twice $4\times 88=372$.
Pupil No. 15. Difficulty with arithmetic, a very slow thinker.
Pupil No. 21. Can't remember basic facts. Has them one day and not the next.
Pupil No. 22. New pupil—slow in all subjects, but excellent worker.
Pupil No. 23. Teacher said: just plain shiftless—missed 7 examples in first ten and 6 in last 5 tests. Some of the errors made by this pupil were $3+1=3$, $3\times 9=28$, $6\times 8=42$, $3\times 9=18$, $4\times 8=36$. These in spite of the fact that he was taught and knew how to get the correct answer to a basic fact by counting.
Pupil No. 29. Had trouble with 3×9.
Pupil No. 30. Slow in all subjects.
Pupil No. 32. This new pupil used casting out 9's[10] to check multiplication but got 15 examples wrong in spite of it on the first 10 tests, but when made to understand the check got only one example wrong in the last 5 tests. The example and its check was:

$$\begin{array}{l} 59=\mathit{14}=5 \\ \underline{\times 34=\ 7} \\ 236 \\ \underline{168\leftarrow} \\ 1916=\mathit{17}=\mathit{8} \end{array} \qquad \begin{array}{r} 5 \\ \underline{\times 7} \\ 35=\mathit{8} \end{array} \qquad \begin{array}{ccc} & 5 & \\ 8 & \times & 8 \\ & 7 & \end{array}$$

Notice that the check did not catch the error in the second partial product. Correctly done it is this way:

$$\begin{array}{l} 59=\mathit{14}=5 \\ \underline{\times 34=\ 7=7} \\ 236 \\ \underline{177} \\ 2006=\mathit{8} \end{array} \qquad \begin{array}{r} 5 \\ \underline{\times 7} \\ 35=\mathit{8} \end{array} \qquad \begin{array}{ccc} & 5 & \\ 8 & \times & 8 \\ & 7 & \end{array}$$

This check when understood consists in counting the beads on the abacus required to represent a number thereon and of taking out the nines if any as: $59=\mathit{14}=\mathit{5}$ means $5+9=14$, $14-9=5$. Elsewhere this writer said, "Proof by nine or casting out nines is difficult to understand arithmetically because it does not represent modern computation. It is descriptive of primitive and ancient calculation with tangible devices like the various forms of the abacus. It may be done mechanically without meaning like punching the keys of a comptometer." In this case we see that proof by nine is a mechanical device and does not check the mental process patterns involved. Manipulation of numbers is not the same as checking required mental computation patterns.[11] The distinction is most important.

Conclusion

From this experiment we may conclude:

1. That it is very difficult to get 100% automatic response even when the possible variables are reduced to a minimum.

2. That process patterns are not few and simple but become quite numerous and complex but show relationships that make example classification possible.

3. Children's errors are not always due to inability to work examples, but to various other psychological factors as:

(1) Undesirable work habits.
(2) Length of a process pattern may cause fatigue.
(3) Pressure due to class competition to finish as quickly as possible may cause a child to substitute unchecked automatic responses for reasoning.
(4) Willingness of some children to substitute automatic manipulation for reasoning.

4. If we consider that the class as a whole worked 4800 examples, and that only 288 or 6% of them were missed, the results though not perfect were very satisfactory.

5. The class tendency showed a slight general higher proficiency for the higher I.Q.'s. (See Illustration H).

6. When process patterns are taught, discovery is good. This class achieved 81% of perfect scores in the first ten tests.

References

1. "Teaching the New Arithmetic," Wilson, Stone & Dalrymple, McGraw Hill Book Co., 1939, pp. 94–95.
2. "How Effective is the Meaning Method?" A Report of the Los Angeles Study, G. H. Miller, THE ARITHMETIC TEACHER, March 1957, p. 45 ff.
3. "A Study of the Process of Transformation," Louis E. Ulrich, Sr., *The Mathematics Teacher*, November 1947, p. 345.
4. "Streamlining Arithmetic," Louis E. Ulrich, Sr., Lyons & Carnahan, Chicago, 1943, p. 291 ff.
5. "Psychological Considerations in the Learning and the Teaching of Arithmetic," Wm. A. Brownell, Tenth Yearbook, N.C.T.M., p. 6 ff.
6. "Introduction," G. T. Busewell, 50 Yearbook, N.S.S.E. Part II, p. 1.
7. "The Problem of Transfer in Arithmetic," James Robert Overman, Tenth Yearbook, N.C.T.M. p. 173.
8. "Streamlining Arithmetic," Chapter 6, p. 81 (See[3] above).
9. "The Meaning of The 'Meaningful' Teaching of Arithmetic," Anita Reiss, *The Elementary School Journal*, September 1944, p. 23.
10. "Twenty-Five Questions on Arithmetic" (See question 13 on Checking, Reasoning, Casting Out Nines) Ben Sueltz, THE ARITHMETIC TEACHER, December 1956, p. 250–251.
11. "Casting Out Nines," Louis E. Ulrich, Sr., THE ARITHMETIC TEACHER, Oct. 1955, p. 77.

EDITOR'S NOTE. Ideally, it would be most gratifying if every adult always were correct in his computations and in his reasoning. A good computing machine in the hands of a competent operator will produce correct results. But human beings are not machines. Mr. Ulrich points out that there are a number of factors that account for a child being correct one day and then being wrong another day with the same exercise. He also shows that computations that may appear to be fairly simple and direct involve many opportunities for error. It is frequently difficult to know precisely how an error was made unless the *thinking-work* of the pupil is recorded at the moment. For example, $3\times 9=27+1=29$ has a number of explanations. The good teacher will seek to locate the nature of error and then proceed with remedial measures. The way this is done is as important as the doing thereof.

Mr. Ulrich's pupils did very well when the total possibilities for error are considered. Out of 4800 examples, 288 or 6% were done incorrectly and each example probably had about 20 opportunities for error. It is apparent that these pupils are taught to think and reason as they do their work. Such learning should be more lasting than mere memory and it gives the individual something to fall back upon when memory fails.

The Number System and the Teacher

(*Continued from page 160*)

EDITOR'S NOTE. Dr. Peters discusses our number system and some of its ramifications in clear concise language which opens this area to many teachers who may not have studied college mathematics. Teachers will note the duality of the "collection" idea and the "line-sequence" idea of numbers and will decide at what levels and for what purposes one is better than the other. This will influence their selection of visual-manipulative aids because an aid should serve a *useful* and *timely* purpose. In a sense the "line-sequence" is a more sophisticated concept and has more possibilities for personal extension by the individual learner. Miss Peters would like all of us to know a great deal of the mathematics behind and beyond the things we teach so that our selection of materials and our development of concepts and principles will enhance the pupils' learning and enable them to proceed independently. A teacher who continues to explore and to consider new vistas in the areas she teaches has the alert qualities which help to make her an interesting individual to both her colleagues and her pupils.

Fraction Concepts Held by Young Children

Agnes G. Gunderson and Ethel Gunderson
Moorhead, Minn.

What can we do to help children gain a better understanding of what fractions are and what they mean? How can we give the child a longer acquaintance period with fractions before he is required to work with the written symbols?

In an attempt to find out what concepts and ideas young children have about fractions, a study was made with a group of children in Grade Two in the Willett's Road School, East Williston, Long Island, N. Y. Individual interviews were had with each of 22 children; two of the 24 children enrolled were absent that week. A separate room was available for the interviews which were held the first week in February, 1957. No attempt was made to get an exhaustive record of each child's knowledge of fractions. The interviews lasted only as long as the children talked spontaneously and easily; some of them kept talking and discovering all the while their replies were jotted down. Ten to fifteen minutes was the average length of each interview.

Introductory Lesson

Because these children had had no work with fractions, the following lesson was taught before beginning the interviews: Each child was given four paper circles 5 inches in diameter. Then these directions were given: Fold one of your circles in the middle. Open it. How many parts does it show? Children answered, "2." What is each part called? "One half." How many halves are there? "2." Take another circle and fold it the same way. Fold it in the middle again. Open it. How many parts does it show? "4." What is each part called? Here the children answered, "a quarter." The term "quarter" is significant; it indicates that the children have had out-of-school experience with fourths (the usual textbook terminology) but in talking about these parts, they use the word "quarters." Through questions such as "How many parts does the circle show?" "What other word could we use besides 'quarters' that would tell there are 4 parts in the circle?," the word "fourths" was elicited. How many fourths does your circle show? "4." What is each part called? "1 fourth." Color 1 fourth. What part of your circle is colored?, "1 quarter, 1 fourth." Color 1 more fourth, one next to the one that is colored. What part of your circle is colored now? Answers were, "2 quarters, 2 fourths, 1 half." Similar work was done with another circle to get eighths. A set of flannel circles was cut at the same time so as to have them for the flannel board. Another day circles marked for thirds were given the children, and thirds and sixths were taught in the same way.

All work was oral—not even the symbols $\frac{1}{2}$, $\frac{1}{4}$ were written on the board.

Fractional parts of circles were used because it is easier for young children to recognize fractional parts of circular wholes than fractional parts of other shapes. A square or rectangle such as a whole sheet of paper, for instance, seems to become smaller wholes when cut in halves, thirds, or fourths.

Interviews

The flannel board and circles (pies) cut in fractional parts were used during the interviews.

In the interviews, not all children were asked the same questions. Some might be asked, "Which is larger 1 *half* of a pie or 1 *fourth* of a pie?" others "Which is more 1 *third* of a pie or 1 *sixth* of a pie?" Always they were asked, "How do you know?" or "How can you be sure?" Besides defending their answers, the children were asked to think aloud as they worked and to tell other things they discovered about these parts. The word "fraction" was not used at all. The children had no difficulty in identifying fractions or fractional parts. When asked, "Give me 1 *third* of a pie; Point to 1 *fourth* of a pie; What is this piece called? Why is it called that?" they would respond, "This is 1 *third* because there are 3 of them; 1 *sixth* because the pie is cut in 6 pieces; 1 *eighth* because there are 8 parts." Only one child seemed a little uncertain but identified all parts correctly, talking as she did so: "I think this is 1 *fourth;* 1 *eighth?*, I might pick a small piece; 1 *third*, I think I'd pick a big piece; 1 *sixth?* I think I'd pick one of these (pointing to pie cut in sixths); 1 *half* (no hesitation), *that* is very big."

(1) Asked "Which of these pies show half a pie?" Kathe said, "All these make half a pie" as she pointed to 4 of the *eighths*, 2 of the *fourths*, 3 of the *sixths*, 1 of the *halves*. Then pointing to the pie cut in *thirds*, she said. "I don't know how to get half of this pie." George, who was later asked the same question said: "Two of these pieces (fourths) add up to one half; 3 *sixths* add up to one half. This doesn't show 1 half (then folding 1 of the *thirds* double) Yes, if you cut 1 of these pieces (thirds) in half it will show half a pie."

(2) How does this pie ⊕ show half a pie? Steven: "2 *fourths* is half a pie; (then added) You need another 2 *fourths* to make a whole pie. 4 *eighths* is half a pie too."

(3) How many *eighths* would I need to have half a pie? Janet replied, "4, because $4+4=8$. $8-4=4$; (then added) You would not need as many *sixths* for 1 half pie as *eighths* because *sixths* are bigger." To this same problem Barbara added, "3 *sixths* of a pie as 1 *half* pie."

Comparing Unit Fractions

While the question named the object as well as the parts, e.g. Which is bigger, 1 half of a pie or 1 third of a pie? the children did not always answer so fully, saying only "1 *half*, 1 *third*, 1 *sixth*."

Questions dealing with unit fractions were given at the beginning of each interview as these seem simpler than other fractions. The responses indicate which fractions were being compared. Many responses were duplicated by several children; only different responses are given here.

Richard said, "1 *half* of a pie is bigger than 1 *third* of a pie." Another child placing 1 *fourth* over 1 *half*, said "1 *half* is bigger than 1 *fourth*."

Stephen: "1 *fourth* is a bigger piece than 1 *sixth* because when you have more pieces they are smaller."

Cydney: 1 *third* of a pie is larger than 1 *fourth;* 1 *fourth* is a little smaller because there are more pieces."

Leslie: "1 *third* is more than 1 *sixth;* it is 1 *sixth* more."

Marian: "You need 2 *sixths* for 1 third."

(4) Which is a bigger piece, 1 *third* of a pie or 1 *eighth* of a pie? Cydney: "One *third* is a bigger piece; (then superimposing) if you took 3 *eighths*, it would be about the same but 1 *eighth* is smaller."

(5) Judy, you may take a piece of pie. (She chose 1 *half*). I'll take this piece (1 *fourth*). Which piece is bigger? Judy: "1 *half* is bigger because it takes 2 of the *fourths* to make 1 *half*."

(6) Which is smaller, 1 *eighth* of a pie or 1 *third* of a pie? Susan: "1 *eighth* is a smaller piece than 1 *third* but the pie divided in *eighths* has more pieces."

(7) Another kind of reasoning was revealed in Marian's answer to Which is more pie, 1 *sixth* of a pie or 1 *fourth* of a pie? 1 *sixth* is more 'cause 1 *sixth* is 6" (holding up 6 fingers; 1 *fourth* is 4 (holding up 4 fingers). Talking on as she superimposed "I think it is more, No, 1 *fourth* is more." Such reasoning is frequently found among children who have difficulty with fractions.

Multiple Fractions

(1) Which is more, 2 *halves* of a pie or 2 *thirds* of a pie? Laurence: 2 *halves* is more; 2 *halves* is a whole pie; 2 *thirds* is not a whole pie."

(2) Which is less, 2 *eighths* of a pie or 2 *thirds* of a pie? Shep: 2 *eighths* is less. *Eighths* are the smallest pieces; *thirds* are next to the largest. 2 *eighths* are only 1 *fourth*, and 2 *thirds* are bigger than 1 *fourth*."

(3) Stephen, you may have 7 *eighths* of a pie. I have 2 *thirds* of a pie. Who has more pie, you or I? Stephen: "3 of my *eighths* about cover 1 *third;* (talking as he superimposes) I'll see; that's the only way to find out. I have about 1 *eighth* more."

(4) Which is more, 2 *thirds* of a pie or 2 *sixths* of a pie? Richard (superimposing): "2 *thirds* is more; 2 *sixths* cover only 1 *third*. If you had 2 *sixths* more they would cover. (Then added) You could make a whole pie with 1 *half*, 1 *third*, and 1 *sixth*."

(5) Which is more, 6 *eighths* of a pie or 3 *fourths* of a pie? Stephen: "2 of these *eighths* make a *quarter;* (then as he covers each fourth with 2 *eighths*) they are the same."

(6) In this problem an indirect solution was used: Which is more, 2 *fourths* of a pie or 2 *sixths* of a pie? Shep: "2 *fourths* is more; 1 *fourth* is one-half of one half; put 2 *fourths* together, it is 1 *half*. 2 *sixths* is not 1 *half;* you need 1 *sixth* more."

(7) Which is more, 1 *half* of a pie or 2 *sixths* of a pie? Robert: "2 *sixths* is more because more pieces; (then superimposing) No, 1 *half* is more."

Whole and Mixed Numbers

These children had no difficulty in carrying out directions such as: Give me 1 whole pie and half a pie; 1 whole pie and 1 third of a pie, and so on. For the whole pie they would select an uncut pie; for the fraction they would take a piece from the pie cut in the appropriate parts. One exception was Cydney who chose the pie cut in halves for the whole pie and 3 of the sixths for half a pie.

(1) Give me 1 and 1 *half* pies. How many *half* pies can I have? Susan: "Cut this whole pie in *half* and you will have 3 *halves*. That is 1 *half* more than a whole pie."

(2) Give me 1 whole pie and 1 *fourth* of a pie. If this pie were cut into *fourths*, how many *fourths* would I have in all? Marian: "1 whole pie is 4 *fourths*, add 1 *fourth*, that is 5 *fourths*." Which is more, 5 *fourths* or 3 *fourths?* "3 *fourths* is only this (pointing to 3 of the *fourths*); 5 *fourths* is 1 more *fourth* than a whole pie. It is 2 *fourths* more or half a pie more than 3 *fourths*."

(3) Betty, you may have a whole pie. (She chose the one cut in eighths.) Will you give me 2 of those pieces? Now, how much pie do I have? Betty: "If you put them together, you'd have 1 *fourth*, but if you don't, it still would be 2 *eighths*. I have more than half a pie left. I have 6 pieces, 6 *eighths* of a pie; that is 2 *eighths* more than half a pie."

(4) If you have 2 whole pies and give away 1 *fourth* of a pie, how much would you have left? Barbara took 2 pies (1 whole pie and the one cut in *fourths*). She then took away 1 of the *fourths* saying: "I would have 1 pie and 3 *fourths* left."

(5) Another problem in subtraction: If you have 8 *eighths* of a pie and give me 3 *eighths*, how much pie will you have left? Mike: "5 *eighths*, that is 1 *eighth* more than *half* a pie."

(6) Mary took 1 *sixth* of a pie, Jill took 1 *sixth*, and Linda 1 *sixth*. How much pie did all of them take? Janet: "3 *sixths*." Barbara: "1 *sixth* and 1 *sixth* and 1*sixth* are 3 *sixths*." What part of the pie was left? "1 *half* of the pie was left."

Using Fractions

(1) Which of these pies has the biggest pieces?

Jill: "The pie cut in *halves*."

How many people could you serve with this pie?

Jill: "Two, you could serve 4 if you cut each piece (half) in *half*."

(2) Which pie will serve the most people?

Shep: "The pie cut in *eighths* serves the most people, but the pie cut in *halves* has the largest pieces."

(3) From which pie could you give 1 serving and have the most pie left?

Stephen: "The pie cut in *eighths* because that has 8 pieces. 8 *eighths* take away 1 *eighth* is 7 *eighths*."

(4) Mrs. Gay wants to serve 5 people. Which pie should she take? Cydney: "She'd probably use this one ⊛. She'd have 1 piece left over." Shep reasoned this way: "There is none cut in 5. She should get the one cut in *sixths*. She would have 1 left over. She could use this one ⊛ (*eighths*) but she'd have 3 pieces left over. It would be better to have only 1 left over."

(5) If Mrs. Gay were serving 7 people, which pie should she take?

Richard: "She'd better take this one ⊛ (*eighths*). There will be 1 piece left over."

Billy: "She wants to serve 7, she must take the *eighths*. 1 *eighth* will be left."

Robert: "To serve 7 people, take the the pie cut in *eighths;* she'll have 1 for herself. To serve 5 people, take the *sixths;* she'll have 1 for herself."

Marian: "To serve 7, choose this one ⊛ (*eighths*). The people who are served this pie get smaller pieces than those served from this pie ⊛ (*sixths*). When you cut more pieces, they are smaller."

(6) Take 2 pies and 1 fourth of a pie. How many people can you serve with pieces of this size ($\frac{1}{4}$)?

Janet, taking 2 whole pies and 1 fourth of a pie, said, "I can serve 4 from this pie and 4 from this pie. $4+4=8$ servings, and 1 more piece. I can serve 9 people."

The following comments reveal the generalizations reached: "If you want many pieces, cut them small. Big pieces make few pieces. The smaller you cut them, the more you get. When you cut many pieces, the pieces get smaller. If you cut pieces smaller you have more. Small pieces, many pieces."

Conclusions

This study indicates that the concept of fractions may well be introduced in Grade Two. These seven-year-olds showed an ability to grasp the meaning of fractions, some showing a deeper insight than others. It is significant, however, that all of them had gained (through their previous experience, including the introductory lesson) the ability to recognize fractional parts, to compare them, and to make the generalization that the more parts a thing is divided into, the smaller the parts become. But, note, the children were able to do this because they had these fractional parts before them—manipulative materials that they could see and superimpose—in other words, they told the answers as they discovered them by using these concrete representations of fractions. One would not say, nor should one expect, that they could have answered these questions without these objects, nor should one wish them to do so.

Note also that the problems were given orally. To hear some one speak of one half of a pie is far different and much more meaningful than merely to see the symbol $\frac{1}{2}$.

The question may arise, "When second graders can do so well, why should fifth graders find fractions difficult?" To take a specific instance: much learning is needed to bridge the gap between discovering with manipulative materials how much is left of 1 whole pie after giving away 1 fourth of it, and working with numbers the example: $1-\frac{1}{4}=?$.

We are familiar with the three stages of learning number concepts: the concrete, the semi-concrete, and the abstract. Many courses of study plan as a year's work the meaning and use of numbers up to 6. Should we not give fraction concepts fully as much time as concepts of whole numbers? Yet when we compare the amount of time allowed for mastering the concepts of 1-place numbers with that of mastering concepts of simple fractions we can see that the child

is rushed through the beginning work in fractions which develops the meaning and into the computation of fractions in their written form. Perhaps, he is given no more than a superficial introduction consisting of a lesson or two with concrete materials such as fractional cut-outs of paper, cloth, or wood. This introduction usually takes place a day or two before work with fractions as outlined in the textbook, which means that the pupils have only a few days' acquaintance with fractions before being asked to work such examples as these: $\frac{2}{6}+\frac{1}{6}=?$ and $\frac{3}{4}-\frac{1}{4}=?$ True, children are older than they were when learning concepts of whole numbers, but maturity in dealing with arithmetic meanings does not come simply by getting two or three years older or through incidental out-of-school activities.

Even though the children interviewed in this study answered correctly almost every question asked, yet they are far from ready to do the computational work as presented in textbooks. A planned systematic program for developing the meaning of fractions is essential as readiness or preparation for working with fraction symbols.

What are the reasons for difficulties with fractions and what help can we give? Up to the time children encounter fractions in written form, they have dealt with whole numbers, each number retaining its intrinsic value; that is: 2 means 2 things; 3 means 1 more than 2; 4 means 1 more than 3 and so on. But in fractions such as $\frac{1}{2}$, $\frac{2}{3}$, etc. the numbers no longer retain their intrinsic value. Heretofore 2 has meant 2 things but in the fraction $\frac{1}{2}$, 2 means that something has been divided into 2 equal parts. Up to now when the child has seen the example: $1+2=?$ he has added 1 and 2. Likewise: $4+4$. Seeing these same numbers in this form: $\frac{1}{4}+\frac{2}{4}=?$ is it strange that he gives the answer as $\frac{3}{8}$? He probably sees two examples in addition with one addition sign and one equal sign serving for both examples.

Then there are other things that may confuse or bewilder the child: the likeness in the sound of words, for instance, yet with such different meanings as seen in the following:

four	4	*fours*	$\begin{array}{r}4\\ \underline{\times 3}\end{array}$
fourth	$\frac{1}{4}$	*fourths*	$\frac{3}{4}$
eight	8	*eights*	$\begin{array}{r}8\\ \underline{\times 5}\end{array}$
eighth	$\frac{1}{8}$	*eighths*	$\frac{5}{8}$

It is small wonder that the child becomes confused. Only through many meaningful experiences with each and all of these symbols and their number names can we expect children to become well acquainted with them and master fraction concepts.

It would seem that an interval of two years or more between a child's first introduction to fractions, as illustrated in these interviews, and the time he is expected to work such examples as: $\frac{4}{8}-\frac{2}{8}=?$ and $\frac{1}{6}+\frac{3}{6}=?$ is not too long.

This acquaintance period should provide for planned systematic work with manipulative and semi-concrete materials. Flannel boards and fractional cut-outs should be available for this work. Other shapes such as squares and rectangles as well as circles may be used. Children may play games with fractions, make up problems using them, and report any uses of fractions in out-of school activities. We should take advantage of every opportunity to teach fractions using apples, candy bars, cookies, birthday cakes and other school treats. Surely there is no better way to make fractions meaningful and delightful to children than to eat them.

In the beginning when writing or recording the work with fractions it is well to use the word rather than the symbol, i.e., writing "3 fourths" rather than $\frac{3}{4}$. Later the denominator as well as the numerator may be expressed in symbols. Fractional parts of a group may also be taught when children are ready, but this is more advanced work than fractional parts of a whole.

It may be wisdom to let the children set

the learning pace. There is danger in hurrying. The teacher's task is to provide the materials and encourage children to make discoveries.

Significant Findings from this Study

1. Young children are interested in and like work with fractions. They showed no frustration, but were confident in their approach to the problems.
2. These children showed a good understanding of fractions when using manipulative materials. These children can obviously profit from planned systematic instruction in the meaning and use of fractions.
3. This study indicates a need for an arithmetic program which introduces systematic work with fractions as early as Grade Two. At this level the teaching must be oral, with manipulative and semi-concrete materials available for children to use.
4. Before such a program can be implemented in our schools, our teachers must realize the need for it; must appreciate that children learn through discovery, not through the memorizing of rules and generalizations;—and perhaps most important of all, teachers must acquire the "know-how" for planning and carrying out such a program. To help them do this is the challenge that faces us as teachers of teachers.

EDITOR'S NOTE. The Misses Gunderson have shown us what they have done with one group of children and this is a challenge for all of us. Perhaps the *how* they did this is as important as the *what* they did. Note how the carefully framed questions in sequence led the children to think, to reason, and to discover. This is far different from giving youngsters the conclusions reached by other people and asking them to memorize. This is more the real essence of learning in the larger concept of child development. Note also that this kind of learning cannot be done via the textbook; it is beyond the possibilities of a text but it is frequently mentioned in the newer manuals which accompany textbooks. Many schools have large quantities of art supplies and a wealth of science equipment but are almost totally lacking in arithmetic materials other than books. It is high time that teachers begin to demand arithmetic supplies. Fortunately, arithmetic supplies are simple and usually are readily available as raw material. Perhaps it is the teacher who is often too short sighted to realize that such common items as strings, pebbles, beads, blocks, cans, paper, and cardboard are useful in the developmental stages of learning arithmetic. As the Misses Gunderson say, this requires some understanding on the part of the teacher.

It would be very interesting to know how the pupils extended the ideas developed in class and in the interviews independently. One of the excellent features of the "discovery method" is that it starts pupils on their own personal roads to learning. But we must allow time for pupils to work and to think.

BOOK REVIEWS

Macmillan Series, Arithmetic Textbooks. Grades 2–8, THE MACMILLAN CO., New York, 1957. (No manuals received.)

Each book is attractive, and with the exception of Grade 2, each book contains a carefully prepared index. The grade level is indicated, not by a numeral, but by the number of letters in the first word of the title of each book.

The series abounds with realistic and interesting illustrations based on children's experiences and colorful diagrams representing mathematical situations.

Interesting, challenging exercises and problems are included for the more mature children in a class. These are indicated with a star and are placed at the end of each page or section.

Drill procedures and tests of basic facts are interspersed throughout all books. Achievement tests are also presented in each book, generally at the end, so that children can evaluate their progress. Basic facts are developed through relationships so that mathematical principles may be seen.

We assume that background material is available in books earlier than the book for Grade 2 or in a teacher's manual for Grade 2. The development in this textbook seems particularly advanced from the beginning.

It is hoped that manuals have been prepared and that these include suggestions and content from earlier grades so that teachers may be able to provide for the mathematical growth of less than "average" pupils.

We do not feel that there is enough emphasis on estimating quantities, or on estimating sums, differences, products and quotients before computing. Procedures for making estimates are limited. It is possible that procedures for estimating are given in the manuals, however.

We hope that the manuals include procedures for teaching measurement since the textbooks do not (probably cannot), adequately develop this area.

Books for Grades 7 and 8 present fundamentals of business practice. Problems are generally related more to adult life situations than to pupil experiences at these levels. Similarly, other aspects of mathematics are developed sequentially on higher levels from book to book.

From the recognition of squares and circles in grade 3, to dealing with plane figures in Grades 7 and 8, each book contributes to a good foundation for geometry. Graphs are introduced in Grade 5 with picture and line graphs and are developed on higher levels through Grade 8. Circle graphs in Grade 8 contribute to understanding of geometry.

The Macmillan series is one which teaches and children will enjoy using.

JACK WERKEMA LOIS KLINE
EVELYN KAHLE HEDWIG HELSTEN
MARIAN MENZEL LUCINDA WRIGHT
ROBERT NUSSBAUM

Refresher Arithmetic, with Practical Applications. Edwin I. Stein. ALLYN AND BACON, INC., 1957, 434+xiv pages. $3.36.

This book has allegedly been designed for use in arithmetic classes at either the junior high school or senior high school level. It is divided into four main parts. The first and longest part, (about one-half of the book), deals with arithmetic, the second with mensurational geometry, the third with "everyday problems," (consumer and business arithmetic), and the fourth with algebra. The section on algebra is very brief, and is concerned chiefly with common formulas.

The first two parts, (on arithmetic and geometry), are subdivided into sections called "Exercises," each of which presents a treatment of one topic. (Examples are: "Rounding off Whole Numbers," "Changing a Fraction to Higher Terms," "Division of Decimals," "Measure of Lengths—Basic Units".) In general, each exercise is developed according to a set pattern. First there is a statement of the aim of the exercise, then a comprehensive statement of the procedures to be used. Then follows a set of of sample solutions, and a list of definitions, (in some exercises). The remainder of each exercise is devoted to a diagnostic test followed by extensive sets of graded practice problems. Non-contextual numerical problems comprise most of the problem sets, but some sets with verbal contexts are included.

This is a "how-to-do-it" textbook. The emphasis is uniformly on techniques and skills of computation. Statements on procedures to be used in solving problems in the various exercises present steps to be followed in completely methodical fashion. No discernible attempt is made to promote understanding of the concepts and principles involved. The reviewer feels that this technique of presentation is most unfortunate, and that it can only promote the too prevalent feeling among pupils that mathematics is merely a series of dull problems to be hacked out.

Even those who prefer a textbook consisting of unembellished directions along with problems for practice will probably find undesirable features in this one. Steps of procedure in each exercise are given with no illustration of steps taken at all. Sample solutions follow statements of procedures, in separate sections. In the exercise on square root, the procedures consume an entire page, and this page must be turned to inspect the sample solutions. In other exercises the procedures and sample solutions are almost as inconveniently arranged.

The wealth of problems in this book may recommend it to some. It could serve a purpose for a person who needs a refresher in elementary mathematics, and who wishes to have simply an inventory of arithmetic processes, with rules and plenty of practice problems. Some arithmetic teachers may wish to have a copy of it for the benefit of its veritable reservoir of problems. But the reviewer questions its usefulness as a basal text in a course at any level. To use it for this purpose would place upon the teacher the entire burden of making mathematics meaningful, alive, and exciting to his pupils; this textbook would be of no help in this. It is hard to conceive of a pupil's enjoying this book.

ROBERT SLAUGH

Some Questionable Arithmetical Practices

HERBERT F. SPITZER
State University of Iowa, Iowa City

MANY IDEAS AND PROCEDURES presented in children's arithmetic books and those used by teachers of arithmetic are seldom examined critically. This situation is due, in the main, to two factors: first, practically all the ideas and procedures used in teaching arithmetic are, for the most part, satisfactory, and second, they have become so familiar to teachers that they are taken for granted. Since there are some arithmetical procedures and some statements of what are purported to be facts that are either erroneous or of doubtful value, it seems worth while to look critically at a few of these questionable practices.

It is hoped that this presentation of a critical evaluation of a few practices will result in the clarification or elimination of such practices and that those interested in arithmetic teaching will, on occasion, examine more critically materials and methods now used in teaching. The five questionable practices follow.

The Year Zero?

1. The use of a zero year as the first year of the Christian Era is one of those arithmetical practices which the writer considers erroneous. The following date line represents the situation:

```
3   2   1   0   1   2   3
|   |   |   |   |   |   |
    B.C.          A.D.
```

This use of zero is an error, just as would be the use of zero in counting the blocks on a street. The first block is number one and not zero. A little thought about the beginning of any year will set one's thinking straight regarding this point. The year 1956 began as soon as midnight was reached on December 31, 1955. Any event occurring after that midnight and before midnight 1957 occurred in 1956. In a like manner, any event occurring after January 1 of the year 754 of the Roman Calendar (the year selected by Dionysius Exiguus as the beginning of the Christian Era) occurred in the first year or the year 1 of Anno Domini (A.D.).

As additional support for the position taken that the Christian Era begins with 1 A.D. and not zero, the following statements are offered. "About the year 532 Dionysius proposed that the epoch of the birth of Christ which he assigned to December 25 A.U.C. 753 should be adopted by Christians and that January 1, 754 become the beginning. Thus A.D. 1 is not the year of the nativity but the first current year after it."[1]

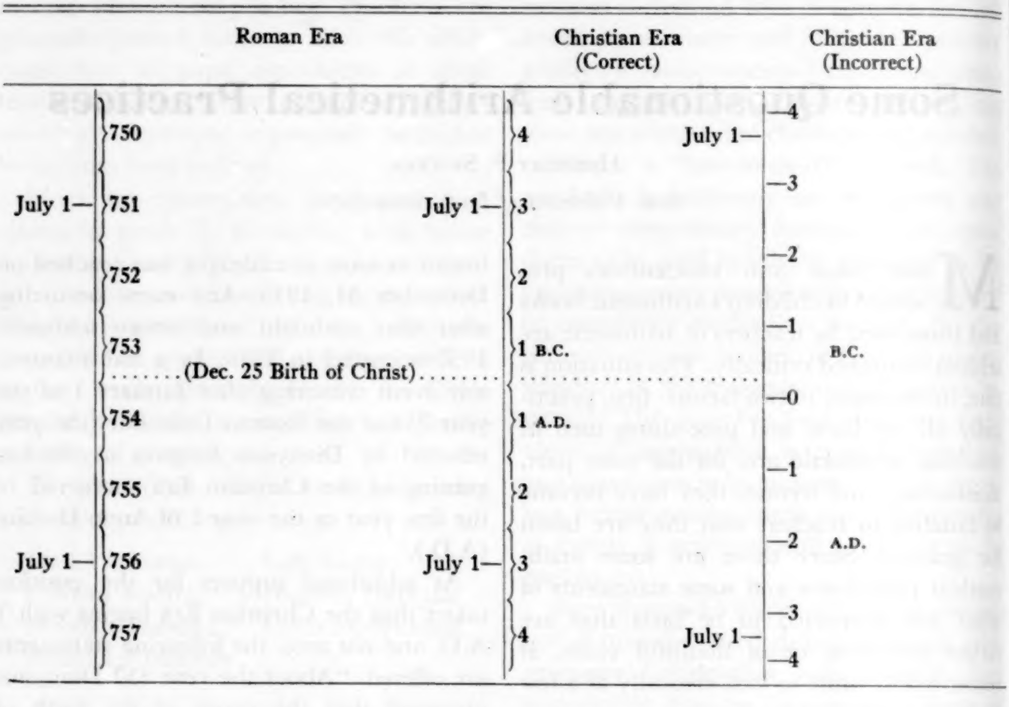
"The year preceding A.D. 1 is called Ante Christian (A.C.) or Before Christ (B.C.). It is to be noted that there is no year '0' as some have imagined intervening between B.C. and A.D."[2]

This use of zero as the first year of the Christian Era is found in a number of children's arithmetic books. This results in children's arriving at incorrect answers in problems involving dates and even to the celebration by classicists of the 2000th birthday of famous men before 2000 years have elapsed. While these are not serious matters, it seems that children in our elementary schools should have an opportunity to learn the correct use of this application of num-

[1] James Hastings, *Encyclopedia of Religion and Ethics*, Vol. III, p. 91. New York: Charles Scribner's Sons, 1911.

[2] *Catholic Encyclopedia*, Vol. 1, page 738. New York: Robert Appleton Co., 1907.

CHART SHOWING TRANSITION FROM ROMAN TO CHRISTIAN CALENDAR



bers. The use of the zero year leads to the perennial arguments about when a century closes. If a century is 100 years, then the first century did not end until the close of the year 100. Therefore, the 20th century did not begin until January 1, 1901.

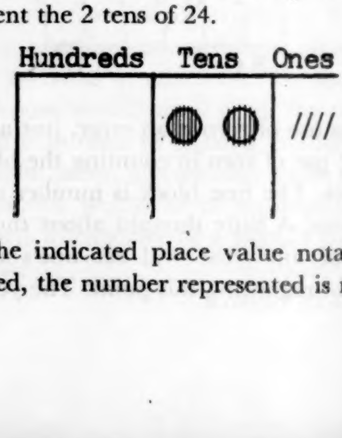
That use of zero as the beginning point for the Christian Era is incorrect can easily be seen by study of the chart showing the Roman and Christian Calendar. By definition, the Christian Era began with the birth of Christ (to be exact, 6 days later). In counting each direction from that point on a time line the first year would, by the rules of counting, be 1 and not zero. As is universally recognized, all time in a year is given the date of the year. Therefore, any event in the year preceding the Birth of Christ occurred in the year 1 B.C. The same type of reasoning would assign 1 A.D. to all events occurring the first year after the Birth of Christ.

As can be seen by use of the chart, the time in years between July 1, 3 B.C., and July 1, 3, A.D., on the correct form of the Christian Calendar is only five years. If the incorrect form (the one using zero) is used, seven years elapse between July 1, 3 B.C., and July 1, 3 A.D. That seven years startles many.

One Marker Means One

2. A second questionable arithmetical practice is in the use of a bundle of tens or of hundreds in place value charts, place value boxes, and similar devices used in representing numbers.

An illustration of this practice is shown below. In the diagram drawing of the place value frame shown, the four single markers placed in the ones position are to represent the 4 ones of 24, and the two bundles of 10 markers in the tens place are to represent the 2 tens of 24.



If the indicated place value notation is followed, the number represented is not the

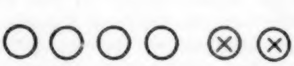
intended 24 but 204. Each bundle is 10. Ten tens equal 100.

Those who propose and use such diagrams as the one illustrated contend that the preceding statement is a misrepresentation of the facts. It is therefore important that the situation be analyzed with care. Such place value frames are used in arithmetic teaching to show that the position of numerals in a number of determines the value of the numeral. That being the case, a single marker in the tens position on the place value chart would have the value of one ten and ten markers would have the value of ten tens or one hundred. The argument is sometimes advanced that placing a bundle of ten in the tens place helps the pupil to see that it is ten. If that is the case, the pupil is seeing something that is of no value as far as the place value aspect of numbers is concerned. If place value frames and similar devices are to be of assistance in teaching the positional value aspect of our notational system, then every marker used, whether it be in the ones, tens, or any other place, should be the same as every other marker. That is the situation that exists when numerals are used to indicate a number. For example, in the number 242, the numeral 2 representing hundreds differs from the numeral 2 representing ones in position only. The use of a defined ten marker in the tens position on the place value frame is then at best only a good example of redundancy.

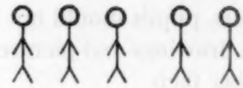
What is a Fact?

3. A third questionable arithmetical practice now used rather extensively is the manner in which drawings are used to show a number fact. The following is an illustration.

(1) "What number fact does this drawing show?"

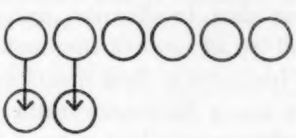


(2) "This picture shows a number story. Write it in the short way."



For illustration 1 the authors intend for the pupil to write $6-2=4$ or $4+2=6$, etc., and for illustration 2 the authors intend for the pupils to write $3+2=5$.

Critical study of the drawing in illustration 1 warrants the following statement: "There are 4 unmarked circles and 2 marked circles." The drawing does not show, without additional explanation, that $6-2=4$ or that $4+2=6$. To show $6-2=4$ with drawings presents some difficult problems. If a "take-away" subtraction situation is used, 6 objects must be shown and then 2 removed. This involves action, something impossible in a drawing. If it is assumed that the 2 marked circles have been removed, then there are not 6 shown. If a "comparison" subtraction situation is used, then the drawing would have to appear somewhat as that shown below. For such a drawing to be understood by pupils, actual work with construction is essential, Then,



for the exercise to be practical, the assignment should indicate that some action has occurred. The following is an illustration: "The drawing shown was used to find the answer to a number question. What is the question and its answer?" Notice that this assignment does not say that a fact is shown by the drawing.

The second illustration, where the picture purported to show that $3+2=5$, is erroneous. The picture shows a group of 3 and a group of 2, or, if seen as one group, a group of 5. The picture does not show that 3 plus 2 equal 5. To show that $3+2=5$ with objects, 10 objects, not 5 objects, are needed. The equality sign indicates that the amounts are equal. The following dot drawing shows $3+2=5$: "• • • + • • = • • • • •"

When drawings and pictures do not show

number facts, pupils should not be expected to use such drawings and pictures to identify basic number facts.

A Definition Should Define

4. A fourth questionable arithmetical procedure is the attempt on the part of many children's textbooks, and of the teachers using such books, to "over-define" arithmetical terms and procedures. For example, many children's textbooks define addition as the putting together of groups and state, further, that only like groups can be put together. A group is, of course, more than one. Then, according to the definition given, $7+1=8$ is not adding. But these same books include all facts involving 1 in their list of basic addition facts. Furthermore, these books and teachers using them will present addition situations where boys and girls are added (of course the sum is labeled "children"), where brown chicks and yellow chicks are added, and so on. The "like groups" in the rule "Only like groups are added" of course refers to the fact that ones should be added to ones, tens to tens, and so on, but that implication may not be understood by all users of the books.

In such instances as those described above, definitions are a hindrance rather than a help. In other words, it is better not to define than to give an erroneous definition.

Useage and Meaning of Words

5. A fifth questionable arithmetical practice is use of language that is inconsistent with the meaning of the term or the procedure used. Perhaps the most glaring of these inconsistent practices is in the basic question used in division. Practically all books and all teachers tell children to read "$3\overline{)12}$" as "How many 3's in 12?" If that question is answered in strict accordance with its meaning, the answer is "none" for there are no 3's in 12, just as there are no nickels in a dime. The 12 is equivalent to four 3's but if the 12 is changed to four 3's, then there is no longer a single group of 12. The number question "How many 3's equal 12?" from the standpoint of meaning is far superior to "How many 3's in 12?" It may be contended that usage has established the use of "in" so well that it may be best to continue its use even though it has definite meaning limitations. However, if "equal" were substituted for "in" in the basic division question, then the pupil identification of division exercises as "guzinto" exercises could hardly occur.

The use of the term "borrow" is another example of an arithmetical situation where the language used is inconsistent with the meaning of the process to which the term refers. This issue has been discussed so often that mention here should be sufficient. It is recommended that "change" be used along with "borrow."

Many books and teachers now use "number story" for "number fact." There is little in such a statement as "$3+2=5$" to justify use of the term "story." Any child who has a good idea of what a story is would be hampered rather than helped by use of "story" to identify an addition fact. On the other hand, pupils who do not know what a story is could hardly benefit from use of the term in connection with an addition fact.

Many other questionable arithmetical practices could be listed, but the five given should be sufficient to call attention to the importance of occasionally taking a critical look at arithmetical practices used by teachers and textbooks.

(*Editor's Note on page 146*)

Learning Principles that Characterize Developmental Mathematics*

Laura K. Eads

Bureau of Curriculum Research, New York City Schools

Developmental mathematics denotes a point of view with respect to pupil learning in the curriculum in mathematics or arithmetic. Some of the characteristics of Developmental Mathematics as this applies to pupil learning are described below.

1. *Pupils are guided in their progress through what are designated as "developmental levels of learning."*
   a. Pupils engage in planned, significant classroom experiences. They are encouraged to focus attention on and to think about the mathematics in their experiences. For example, a child thinks about ways to share an apple with another child, or to cut the cake so that 4 or 6 or 8 children may have equal shares. Pupils see groups of coins and estimate the amount of money collected today. They judge comparative lengths of shelf and paper, comparative capacities of aquarium and measuring jar, comparative temperatures today and yesterday.
   b. Pupils use materials to represent specified mathematical situations. They are given time and opportunity to make mathematical discoveries. They discover groups within 5 discs, or the number of dimes and pennies in the combination of 34 cents and 25 cents.

Pupils are helped to see and think out relationships between 3 threes and 4 threes, between 3 threes and 6 threes, between 4 threes and 8 threes. They see the doubles within 16 and are helped to relate near-doubles to these: 8 and 8, 8 and 9, 9 and 8, 8 and 7, 7 and 8. Pupils use fraction materials and discover that a fourth is half of a half, and that a sixth is half of a third. They see that 4 times $2\frac{1}{2}$ is 8 wholes (4 twos) plus 4 halves, that the 4 halves may be exchanged for 2 wholes, that the produce is 10 wholes.

   c. Pupils are encouraged to think out several ways to arrive at a particular sum, or difference, or product, or quotient. The conventional or standard way to compute, using paper and pencil, is seen as but one way—sometimes a short-cut way but often not. For $4\frac{1}{8}\times 32\frac{1}{2}$, one of the ways to arrive at the product is to multiply "mentally" from left to right. This is really a short-cut way: 4 thirties (120) plus 4 twos (128) plus 4 halves (130), plus one-eighth of 32 (134) plus one-eighth of one-half ($134\frac{1}{16}$).
   d. **Drill and practice exercises are so devised that pupils continue to use mathematical relationships and principles. Drill and practice are actually part of the development of mathematical concepts, rather than repetitive exercises following the learning of concepts.**

For each mathematics topic, multiplication with fractions, for example, pupils proceed from things to mathematical relationships and principles, from classroom experiences to mathematical insight, from seeing mathematics in life situations to seeing mathematics abstractly.

2. *Pupils are guided in their progress from less mature to more mature mathematical concepts and meanings.*
   a. Mathematical concepts are never completely developed. For example, the young child's concept of *2* may be described as "both"—both shoes, both mittens, etc. In school he learns about 2 things, and that 2 things are less than 3 things or 4 things or 5 things. Later he learns that 2 is a number in a continuum, that positive 2 is in one direction from zero and negative 2 is in the opposite direction. When he learns the binary number system he learns a new numeral for the number *2* and writes: $1+1=10$.

      As pupils grow in maturity more and more complex concepts and meanings are developed. A particular topic is not considered a Grade 5 topic, for example, where the so-called "average" pupils will learn it. In the grades before Grade 5, concepts for the topic have been developing. These continue to be developed in Grade 5 and after Grade 5, on progressively higher levels of maturity.
   b. Pupils are helped to learn mathematics sequentially and systematically. They learn the meanings and structure of our number system sequentially over a period of years, from simpler to more complex concepts—non-numerical concepts (many, more, etc.), then the numbers as sizes (*4* as 4 things), then numbers in series (1-2-3-4, etc.), then numbers as tens, and finally our decimal system of numeration (our reading and writing numbers).

* Presented at the meeting of the American Psychological Association in New York City.

Pupils use their understanding of our number system to learn operations with whole numbers, fractions, and decimals. They learn the relationships of one operation to another—counting leading to addition and then to subtraction, addition leading to multiplication, addition and subtraction as inverse operations, multiplication and division as inverse operations, division as successive subtractions.

Pupils build on earlier steps in the mathematical structure as more advanced steps are developed. Earlier steps in the mathematical system are developed very gradually so that understanding is assured, so that later steps are more likely to be derived from earlier steps. This makes for economy of learning.

c. Pupils are given time and opportunity to think out solutions. They apply the meanings they understand to each new situation. This is in contrast to being shown how to do each step in each operation and then memorizing the endless facts and procedures.

Problem solving is conceived of as the thinking through of mathematical relationships. Problem solving is not reading; it is thinking.

Drill and practice involve thinking. Automatic response to number facts is interprete as a final step in seeing relationships more and more rapidly.

d. Pupils' experiences are continually drawn on to give reality and significance to the mathematics being emphasized. But experiences do not determine the emphases or sequences to be developed.

Mathematical emphases and sequences are determined by the complexity of concepts involved. Multiplying or dividing, using the standard written steps, involve extremely complicated concepts and are therefore introduced after simpler concepts of multiplication and division have been developed for a period of years.

Pupils proceed systematically from topics involving simpler concepts to topics involving more complex concepts. As each topic is developed throughout the years concepts grow in complexity and become clearer and more precise. Understanding and insight into relationships and procedures for a particular topic facilitate the learning of more advanced relationships and procedures for the topic. Rapid and accurate computation is facilitated as relationships continue to be emphasized.

3. *Differences among pupils are provided for as they proceed through "developmental levels of learning," and as they grow in development from less mature to more mature mathematical concepts and meanings.*

a. Within a particular class the pupils vary in ability to deal with mathematical abstractions for the topic being developed. The teacher recognizes this. In developing addition with 2-place numbers, for example, the teacher plans to have some pupils use dimes and pennies on Place Value cards to represent the experience situation and to arrive at the sum, while other children arrive at this sum and other similar sums "mentally."

Thus some pupils think referring to experience materials, others think referring to representative materials, others think referring only to number symbols.

b. Pupils vary in the degree of understanding developed during a particular lesson or series of lessons. The teacher does not think: "I taught. The pupils learned." The term "grade work" takes on new meaning. The teacher plans each day to develop concepts on progressively higher levels starting with mathematics for the topic introduced in lower grades.

c. Particular pupils vary in their understanding of different topics and in readiness for successive steps for different topics. This is sometimes the result of their having learned immature methods that "work" temporarily. For example, some primary grade teachers emphasize counting-by-ones instead of recognition of groups without counting. Some pupils use this over-learned counting-by-ones method to arrive at sums or remainders. These pupils make rapid progress for some time. Then, inevitably, the immature system, too well established, interferes with the learning of more difficult additions and subtractions, and, later, with the learning of multiplication and divisions.

The teacher studies the sequences for developing each topic beyond the level of the grade to make sure that the procedures emphasized in the grade will lead to more and more mature levels of thinking in later grades.

d. The teacher provides time in each lesson for the evaluation of levels of thinking within the class. The teacher asks questions that require thinking and then listens to slow pupils, average pupils, and bright pupils as they express their thinking. Thus the teacher learns when and from whom to accept and expect an immature response, and when and from whom to elicit a more mature response.

e. The teacher is alert for evidences that the content being presented is beyond the mathematical levels of particular pupils. Pupils who consistently fail in thinking out solutions may cease thinking mathematically. These pupils, then grasp at phrases heard from others, and memorize and repeat these, hoping that this will help them "save face."

The teacher asks pupils to make estimates before computing. Unreasonable estimates may be evidences of lack of readiness for the task. The teacher interprets errors and "foolish mistakes" also as possible lack of readiness for the task.

Pupils who are not ready for the content

being presented to the group as a whole are presented with earlier steps in the sequence for the particular topic. The teacher does not spend time trying to "bring slow pupils up to grade." Instead these pupils are helped to progress only as rapidly as they can continue to think out mathematical relationships. Bright pupils, on the other hand are presented with tasks that challenge them on their levels of maturity.

The teacher evaluates pupil learning continually so that pupils will develop concepts and think mathematically at their individual levels of maturity. The teacher plans to develop each topic sequentially beginning with early-level concepts introduced in earlier grades and proceeding to more and more mature levels of thinking.

4. *Pupils participate in planning and conducting learning experiences and in evaluating achievement on their levels of maturity.*
   a. Pupils participate in experiences from which mathematical situations are derived. These experiences vary from blockbuilding and and house play in Grade One to conducting a classroom store or bazaar in later grades.
   b. Pupils participate in planning where and when to store and use concrete materials for mathematical emphasis. Pupils mature enough for independent activities use such materials individually, in pairs, or in small groups to solve mathematical problems.
   c. Pupils participate in preparing and conducting drill and practice exercises, emphasizing the relationships and principles previously developed. They also participate in preparing and conducting tests to find out whether responses are more or less automatic.
   d. Pupils participate in keeping records of their achievement. These may be kept in individual folders and referred to when necessary by pupil and teacher.

The teacher allocates activities so that pupils performing these are challenged at their levels of maturity. Thus pupils who still need experiences involving counting by groups are those selected to count the money. Pupils who still need to write number symbols are selected to make signs for the classroom store or to prepare drill cards.

Program of Curriculum Research

The foregoing learning principles characterizing Developmental Mathematics evolved as a program of curriculum research developed. Some phases of the research follow:

1. An experimental study of curriculum procedures were gifted children in homogeneous and heterogeneous groups revealed that gifted children and their teachers were dissatisfied with their progress in arithmetic compared with their progress in reading, science, social studies, art, etc. Gifted children were interviewed. Children from every class said that they would like to understand what they were doing in arithmetic.
2. A study of mathematical concepts held by gifted children in primary grades revealed that these children had little or no understanding of mathematical vocabulary they were using, for example: plus, minus, equals, etc. Number concepts were generally vague.
3. Experimental research in selected primary grade classes in selected schools was initiated. The mathematical thinking of children at varying levels of ability was studied—of individual children, children in small groups, children in class groups. Experimental teachers developed procedures for teaching mathematical concepts. Memorization of addition and subtraction facts was not a conscious goal.

   Children in experimental classes learned addition and subtraction facts much more rapidly, and without errors or finger counting, than children in classes where the predominant method was daily drill using flash cards. Children in experimental classes became more and more interested in learning arithmetic, while the reverse was true in control classes. In one class the children said they "ate their arithmetic," referring to the many sharing-of-food and party experiences in their class.

   Experimental research in selected classes was extended each year to include the next higher grade, dropping the lowest grade. It started with Grades 1–3, and was continued until Grades 7–9 were involved.
4. Experimental tryout of meaningful procedures was initiated in Grade One on a city-wide basis, after one year of experimental research in Grade One classes. Each succeeding year the next higher grade was involved on a city-wide basis.

   Classroom teachers, designated Elementary Mathematics Coordinators, were trained to help other teachers in their districts. There are 31 of these for the 20,000 elementary school teachers.
5. A workshop for assistantants-to-principals and principals was organized. These met one morning a week for 3 years. The supervisors worked experimentally with the teachers in their schools.

   Cooperatively, the teachers and workshop participants developed experiences, materials, procedures, content, drill devices, procedures for evaluating pupil learning, etc. Workshop participants organized and led short workshops with principals throughout the city.
6. Materials for bulletins for teachers are planned in committees. Outlines are sent to teachers throughout the city and suggestions for content, materials, and procedures are solicited. Mimeographed materials are prepared. These are sent to teachers and supervisors throughout the city for tryout, criticism, and review. Again materials are solicited.

Again, experimental research in selected classes is initiated. Teachers try out the procedures suggested in the mimeographed materials. The thinking of children at varying levels of ability is studied. Materials continue to be revised until classroom teachers and supervisors are reasonably certain that a bulletin will be effective if used by teachers generally.

The program of meaningful arithmetic that has been developing in the elementary schools of New York City is based on curriculum research which may be designated as action research. Teachers and supervisors have been and continue to participate in all phases of the research: formulating problems, developing hypotheses, determining procedures, evaluating procedures, testing hypotheses, preparing materials, testing materials, and implementing procedures and materials.

EDITOR'S NOTE. Dr. Eads has given many of the steps and procedures that characterize the arithmetic program of New York City. They call this "Developmental Mathematics" because its primary aim is the development of mathematical understanding and the ability to sense and to use mathematics in the child's environment. To plan and carry forth such a program in a large city is a tremendous undertaking. Dr. Eads and the school people of New York City are to be congratulated on this move to revitalize the teaching and learning of arithmetic.

Teaching Concepts of Linear Measurement

JAN JENKINS
Bethany College, Lindsborg, Kansas

THAT NEITHER SCHOOL nor life experiences have made linear measure meaningful to many adults is evidenced by their hazy notions of distances such as those measured in inches, feet, yards, and rods. These same hazy notions formed in early school days, may later in the lives of the individuals, become obstacles in the building of the concepts of perimeter, area, and volume.

To prevent this kind of confusion in the minds of a class, the arithmetic teacher should provide muscular experiences to help make the learning memorable. In so doing, it is important the teacher should plan concept building in proper sequence.

The first idea to emphasize in connection with the teaching of linear measure is that the measuring is done along a line. The pupil should have many experiences moving his fingers along lines, speaking of the exact lengths as he does it; for example: "This line is two inches long." He should also walk along lines on the floor or ground, again speaking of the lengths of the distances as he does so. Often, the following statement should be made: "We are measuring along a line." At the time the teacher thinks appropriate—which should be previous to the child's experience with the word "linear" in the textbook—the teacher should impress on the pupil's mind the following idea: "This kind of measure is called *linear measure*. The word 'linear' comes from the word 'line.'" The two words should be written on the blackboard, with their common letters underlined.

Building the concept of the length of an inch or more requires much estimation followed by actual measurement of the distances. A narrow strip of oak tag an inch long and perhaps a strip two inches long should be used in doing this measuring. The pupil should estimate the lengths of his fingers, the lengths and widths of books and desks, and then measure them with his one-inch or two-inch measure. Some of the more advanced pupils should be able to use three-inch, four-inch, and perhaps six-inch long pieces of oak tag for measuring.

To become familiar with the length of the

foot, the child may make a foot measure of oak tag, measuring it with his one-inch strip to verify the fact that it is 12 inches long. At first, he should not mark off the foot measure in inches. After estimating in terms of feet, such lengths as those of the dimensions of a picture, the desk, etc., the pupil should do the actual measuring with the foot strip. To aid in estimation, there should be many exercises in which each pupil holds his hands an estimated distance of a foot apart; other pupils then measure the distance to check his accuracy.

At a later stage, the foot should be marked into inches and used for measuring to the nearest inch. The pupil should not be permitted to use a ruler marked into units smaller than the inch until he has some knowledge of halves, fourths, etc.

The same general procedure should be used for making meaningful the length of the yard. The yard lengths of oak tag which the child uses at first should have no markings. The child estimates the dimensions of the hall and the classroom to the nearest yard. Eventually the oak tag should be marked into feet and into inches, both of these being done by the pupil himself. He thus discovers for himself that 36 inches or 3 feet equal one yard. Two children should attempt many times to stand a yard apart. Such exercises, when followed by checking, will result in growth of the pupils' concept of the length of a yard.

Rod-long heavy cords should be measured with the foot and yard measures and the discoveries recorded. Estimations in rods of the dimensions of the hall, of the classroom, and the distance between two points on the sidewalk plus actual measurement build the pupil's skills in judging distance. Two children should practice standing apart the distance of a rod until their estimate becomes fairly accurate. At playtime, the class should estimate in rods the length and width of the schoolground or of some other plot of ground. Placing all the rod measures the class has made end to end in a straight line will give the class perspective on the length of the total number of rods.

This measuring, whenever possible, should be made along a line which is already in existence or one which is made by a string or a mark of some kind. Thus the child will again and again engage in *line* measuring.

Estimating and measuring both dimensions of rectangular pictures, book covers, desk tops, tablet covers, classrooms, and halls is building readiness for the understanding of perimeter and should be encouraged.

At the end of each discovery in this activity, a placard carrying the statement of the discovered fact should be exhibited on the wall. Beside it should be a cord or narrow strip of oak tag to show the fact which is on the placard. Beside the statement "12 inches = 1 foot" should be placed both the inch and foot measures. Beside the statement" $16\frac{1}{2}$ ft. = 1 rod," should be placed the rod and the foot measures. The yard and foot measures should hang beside the "3 ft. = 1 yd." placard. Due to the lack of height of the schoolroom, the rod measure will have to be placed parallel to the floor. This restriction should provide a learning situation for the pupils.

Through much estimation and actual measuring of distances which they follow with their fingers or with their feet, children discover what various units of measure really are. Thus their learnings become memorable and clear.

EDITOR'S NOTE. Miss Jenkins asks us to give significance to our work with linear measures so that children will not only learn but will tend to remember. The role of physical experience and impressions gained therefrom is featured. Similar experiences are desirable when surface measure and the measurement of volume are being introduced. To make arithmetic meaningful, understandable, and useful to pupils is our goal. Let us provide the necessary experiences so that the goal may be reached.

Developing Flexibility of Thinking and Performance*

J. Fred Weaver
Boston University School of Education

We all are familiar with numerous changes that have taken place, and are taking place, in connection with programs of arithmetic instruction in our elementary schools. These changes are reflected in various aspects or phases of the teaching-learning process, beginning with underlying instructional objectives. Here we find that, among other things, increased attention is being devoted to the development of what we sometimes refer to as "power in quantitative thinking."

Without question, the attainment of this desirable objective is facilitated by the current emphasis upon mathematically meaningful instruction. However, all too often even this emphasis seemingly fails to produce the anticipated degree of power in quantitative thinking. This is most likely to be the case when attention has been focused too narrowly or too exclusively upon the development of conventional or stereotyped patterns of thinking and performance, meaningful though they may be.

Value in the Conventional

Do not misunderstand. It is desirable—even necessary—that children develop and become proficient with more or less conventional or standard ways of thinking and performing in a wide variety of quantitative situations. Much of our instructional effort must be directed toward this end.

Let us look to the following two examples for illustrative purposes.

$$\text{(A)}\quad \begin{array}{r} 28 \\ \times 4 \\ \hline 112 \end{array} \qquad\qquad \text{(B)}\quad \begin{array}{r} 4\frac{7}{8} \\ +3\frac{5}{8} \\ \hline \end{array}$$

* Paper presented at the Thirty-Fifth Annual Meeting, National Council of Teachers of Mathematics (Philadelphia, March 30, 1957).

In each of these instances we attempt to develop patterns of thinking and performance which are not only mathematically meaningful, but which also are rather conventionalized or standardized. These conventional thinking patterns and related algorisms are so familiar that there is no need to mention them specifically here.

It is not the writer's intention to question the desirability of such standardized patterns of thinking and performance. They have recognized value, and their development rightfully becomes an important phase of our instructional program in arithmetic. However, there is good reason to question strongly—and even deplore—any tendencies we may have to emphasize conventional or standard patterns to the virtual exclusion of other modes of thought and related performance.

Value in Flexibility

One of the attributes of power in quantitative thinking is *flexibility*—i.e., the ability to think about, and react to, a quantitative situation in a variety of ways. The value of this, of course, does not lie in the idea of flexibility *per se*. Rather, value lies in the ability to select from several alternative modes of thinking or performance the one or so that would be most helpful or effective under given quantitative circumstances. The conventional or standard pattern is not necessarily the "best way" in all situations.

Let us return to our two illustrative examples and look briefly at the idea of flexibility in operation there. For our purposes here it matters not whether these examples are to be worked orally or in written form, or whether they appeared originally in so-called abstract form or were derived from a

verbal problem situation. The basic idea to be illustrated is fundamentally the same in any event.

Two likely alternative thinking patterns that might be used with

$$\text{Example (A)} \quad \begin{array}{r} 28 \\ \times 4 \\ \hline \end{array}$$

are indicated below.

First Pattern	*Second Pattern*
4 twenties are 80.	4 times 2 tens is 8 tens or 80.
4 eights are 32.	4 times 8 ones is 32 ones or 32.
80 and 32 are 112.	80 and 32 are 112.

Either pattern of thinking might be recorded in alternate symbolic forms. For instance, the first thinking pattern might have been recorded in either of these two forms.

First Form / *Second Form*

$$\begin{array}{r} 20 \\ \times 4 \\ \hline 80 \end{array} \qquad \begin{array}{r} 8 \\ \times 4 \\ \hline 32 \end{array} \qquad \begin{array}{r} 80 \\ +32 \\ \hline 112 \end{array} \qquad\qquad \begin{array}{r} 28 \\ \times 4 \\ \hline 80 \\ +32 \\ \hline 112 \end{array}$$

Similarly, an alternative thinking pattern and related symbolic record for

$$\text{Example (B)} \quad \begin{array}{r} 4\frac{7}{8} \\ +3\frac{5}{8} \\ \hline \end{array}$$

well might have been as follows:

Thinking Pattern	*Symbolic Record*
4 and 3 are 7.	$4+3=7.$
$\frac{7}{8}$ and $\frac{5}{8}$ are $\frac{12}{8}$, or $1\frac{4}{8}$, or $1\frac{1}{2}$.	$\frac{7}{8}+\frac{5}{8}=\frac{12}{8}=1\frac{4}{8}=1\frac{1}{2}.$
7 and $1\frac{1}{2}$ are $8\frac{1}{2}$.	$7+1\frac{1}{2}=8\frac{1}{2}.$

Other thinking patterns and symbolic records are quite likely with Example (B) just as with Example (A). Some of these will be indicated later.

Too often we take a rather unfortunate attitude or point of view toward these alternative modes of thought and performance. We tend to look upon them as unimportant, as inconsequential, as unnecessary, or even as undesirable. We must, however, come to sense the true value and importance of flexibility in relation to the development of power in quantitative thinking. Just as we see the place of conventional or standard patterns of thinking and performance in our instructional programs, so we must become convinced of the necessity for developing alternative modes of thought and behavior in quantitative situations. It should be obvious that unless we recognize value in this idea of flexibility, we will do little if anything to provide for its systematic development in our teaching.

At this point it may be valuable to inject a related significant observation, even though it will be treated only parenthetically without any extended discussion. The parents of many children now in our schools were subjected to an arithmetic program in their day which, among other things, virtually glorified rigid, stereotyped thinking and performance. Emphasis was placed upon *THE* way to think and act in specific quantitative situations. These parents commonly expect the same for their children in the arithmetic program today and transmit this feeling, to one degree or another, to the children themselves. Thus, in addition to our own recognition of value in the development of flexible patterns of thinking and performance as a vital part of an effective arithmetic program, we also face the necessity of helping both children and parents come to a realization of the importance of this instructional emphasis.

Acceptance and Recognition of Flexibility

Now as we look to some positive things we can do that will aid children in developing this desired flexibility of thinking and performance in quantitative situations, we must realize that children often develop a marked degree of flexibility more or less on their own initiative. In some instances this occurs *in spite of* our instructional program rather than because of it. In other more fortunate instances this flexibility is at least an indirect if not a direct consequence of the meaningful nature of our teaching-learning activities and experiences.

In either event, if we do nothing else there are at least two positive steps we can take that will be of value to children who show signs of developing flexibility largely on their own initiative. For one thing, we willingly can accept and take recognition of the various alternative or "unconventional" modes of thought which children often discover and use effectively. All too frequently, however, we are too prone to do just the opposite. It truly is a tragedy when a child works through a quantitative problem situation successfully in his own way and then is told, in effect: "Well, your answer is right but I'll have to mark the problem wrong—(or take off some credit)—because you didn't work it the *right way*." (!)

The writer is quick to recognize that there are times in our instructional work when we definitely want children to have experience or practice with a specific mode of thinking or performance. Even then, however, the utter rejection of an alternative pattern of thought or written algorism is not just inexcusable; it is quantitatively criminal! Unless we properly accept and take recognition of these so-called unconventional thinking patterns when they are in evidence, we will but suppress and stifle the tendency which children have to develop some degree of flexibility on their own initiative.

It is important that we accept and take recognition of demonstrated flexibility on the part of all children. However, we need be especially concerned about what may happen if we fail to do so with children whose flexibility of thinking is in evidence to a marked degree. An actual illustration may serve to emphasize this point.

A second grader, Mike, was along with the writer in a grocery store. When two jars of peanut butter, each priced at 49¢, were put in the shopping basket, Mike said: "I can tell you how much you'll have to pay for both of them . . . 98¢." When the writer asked Mike how he figured that out, he said: "Well, 40 and 40 are 80, and 9 and 9 are 18. 80 and 10 are 90, and 8 more are 98." The writer then mentioned that 49¢ was close to an amount of money that Mike likely heard about rather often, and wondered if that might help Mike "prove" that 98¢ was right by figuring it out another way. After a few moments Mike said: "Oh, I get it! 49¢ is almost 50¢, and 50¢ and 50¢ make 1 dollar. So, I take 1¢ away and get 99¢. . . . Oh, oh! I forgot to take the other penny away, so that's 98¢."

The writer truly is fearful of what will happen to Mike's potential unless he is in a classroom where his marked degree of flexibility of thinking will be accepted and given adequate recognition.

Encouragement of Flexibility

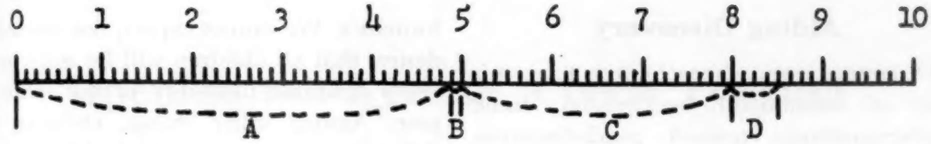
With children who shows signs of developing flexibility of thinking and performance, we naturally can and should do more than just accept and take recognition of demonstrated flexibility—desirable as this may be. It is a simple but effective thing to take a second step and specifically encourage such children to try to discover and use alternative modes of thought and procedure.

For example, when children are working with representative materials in arithmetic we can encourage attempts at manipulation in a variety of ways. These lead to, or result from, variations in thinking patterns which can and should be related to corresponding symbolic records. This may be illustrated with an example cited previously:

$$\begin{array}{r} 4\frac{7}{8} \\ +3\frac{5}{8} \\ \hline \end{array}$$

One alternative thinking pattern and related symbolic record already has been mentioned for this example. If familiar cut-out fractional parts were used, the appropriate manipulation would correspond to the pattern of thinking developed: The 4 wholes and the 3 wholes would be combined first, giving 7 wholes. Then the $\frac{7}{8}$ and $\frac{5}{8}$ would be combined, giving 12/8. This would be changed to $1\frac{4}{8}$. When the 1 is combined with the 7 wholes and the $\frac{4}{8}$ is changed to $\frac{1}{2}$, the sum of $8\frac{1}{2}$ is found.

Let us suppose that a child used a "num-



(A) Go over to $4\frac{7}{8}$.
(B) $4\frac{7}{8}$ and $\frac{1}{8}$ more are 5. $4\frac{7}{8}+\frac{1}{8}=5$.
(C) 5 and 3 more are 8. $5+3=8$.
(D) 8 and $\frac{4}{8}$ more are $8\frac{4}{8}$ or $8\frac{1}{2}$. $8+\frac{4}{8}=8\frac{4}{8}=8\frac{1}{2}$.

ber line" to perform the same addition. Then his "manipulation" and thinking might have been of a different nature, somewhat as indicated above.

This point need not be labored with further illustration. It is clear that we profitably can encourage children to look for alternative ways of working with representative materials, to look for alternative ways of thinking, and to look for alternative ways of recording manipulation and thinking in written symbolic form.

Summarizing Thus Far

It may be helpful to pause briefly to recall the points of view emphasized thus far. At the outset attention was directed to the necessity of sensing value in the development of flexibility of thinking and performance as an instructional objective. Then, noting that children often develop some degree of flexibility on their own initiative, mention was made of two simple but effective things we can do to take fuller advantage of this fact. First, we can accept and take recognition of demonstrated flexibility. Second, we can encourage children specifically to try to find alternative ways of thinking and performing.

Admittedly, these suggestions place the initiative for developing flexibility more in the hands of the child than in the hands of the teacher. Obviously, the teacher can and should play a more active role in this development than has been indicated thus far. We now must look to some of the directions the teacher may take in this regard.

Appropriate Meaningful Instruction

Most importantly, the teacher can facilitate the development of flexibility of thinking and performance by providing mathematically meaningful instruction which emphasizes number relations, basic laws or principles of operation with numbers, and the like. To illustrate, let us look back to an example used previously and one of the thinking patterns that might have been used:

$$\begin{array}{r} 28 \\ \times 4 \\ \hline \end{array}$$

"I know that 4 twenty-fives are 100. 28 is 3 more than 25, and 4 threes are 12. So, the product is 100 and 12 or 112."

Some of the crucial aspects of this pattern of thinking are seen in things such as: (1) quickly sensing that 4 and 25 are factors of 100, or that $4\times25=100$; (2) quickly sensing 28 in relation to 25: i.e., as 3 more than 25, or as $25+3$; and (3) sensing an application of the distributive principle: $4\times28=4\ (25+3)=(4\times25)+(4\times3)$.

If our instruction has been meaningful in the sense that quantitative relationships such as these have been emphasized, then we will have done much to make thinking of this nature possible. In the last analysis our major purpose should not be that of teaching alternative patterns of thinking and performance specifically as such, although upon occasion this may seem desirable. Rather, our major purpose should be that of setting the stage effectively so that these alternative modes of thinking and performance may be discovered more readily by the children themselves. This is the more fruitful kind of learning and is facilitated best through meaningful instruction of the nature just illustrated.

Aiding Discovery

Of course, even when children have this kind of meaningful background, some teacher-guidance toward pupil-discovery often is advantageous or even necessary. A suggestion or hint here, a question there—if injected appropriately, these help to lead children along the road to discovery without actually imposing a pattern of thinking upon them.

Think back to the writer's little second-grade friend, Mike. He was helped by just such a thing. Or think of a youngster confronted with the multiplication, 4×28. The teacher's suggestion that if he knows how much 4 twenty-fives are, he might be able to find 4×28 easily—a suggestion like that may open the door to the pattern of thinking mentioned in the previous section.

Children enjoy and profit from sharing these discoveries with other youngsters. We very definitely will want to provide opportunity for this in a systematic way. As a part of this sharing, we also will want children to consider and discuss the comparative merits or advantages of alternative patterns of thinking and performance that have been suggested. Not all may appear to be equally helpful or effective.

In this connection we *must* keep in mind that "helpfulness" and "effectiveness" are relative rather than absolute matters. What is "good," shall we say, for one child is not necessarily just as "good" for another. Any consideration of the helpfulness or effectiveness of alternative patterns of thinking and performance must be viewed in relation to the child using them, not in relation to any absolute standard of merit or worth.

Individual Differences

This leads to a final point to be emphasized in the present discussion. It is a most important consideration, although it can be touched upon only briefly here.

If ever we need to be mindful of individual differences among children in regard to arithmetic abilities, it is in connection with this idea of flexibility of thinking and performance. We cannot expect, nor should we desire, that all children will be able to develop desirable flexibility to the same degree. Among other things, childern will differ in the extent to which this ability is attained and in the maturity levels at which this ability is operative or functional. We must anticipate and accept such a fact.

We must recognize also that a given amount of emphasis upon flexibility may be fine at one time but not at another; that it may be helpful to one child but confusing upon occasion to another. We must realize that children differ in the extent to which flexibility most profitably can be emphasized before or after the development of conventional thinking patterns and standard algorisms. And we must recognize that a given child's ability to develop flexibility may vary as he progresses from one phase of sequential learning in arithmetic to another.

The Challenge We Face

The important thing for us to keep in mind is that virtually *every* child can develop *some* degree of flexibility of thinking and performance *to advantage*. Our challenge lies in providing instructional opportunities at all grade levels that will enable each child to develop this ability to the point of optimum value from the standpoint of his own potential. It is hoped that the suggestions advanced in the present discussion may enable us to accept and meet this challenge more effectively.

EDITOR'S NOTE. Flexibility of thinking with and about number and quantity as well as of operation with them is a mark of developmental education which distinguishes it and sets it at a level above the older "trade-school" concept in which children were given a pattern of operation to follow. Flexibility is a characteristic of learning in the "discovery method." It is frequently amazing the routes of reasoning through which a child will pass in reaching a conclusion. They may not be the most direct routes nor the most elegant but they are *his* routes and should not be ridiculed automatically by an adult who has a "better method." Give the child an opportunity to think, to refine his steps, and to advance to a higher level. As Dr. Weaver says, give him instruction which encourages flexibility. But to do this a teacher must have a favorable mind set.

Opening the Eyes of a New Teacher

SUZANNE FLEMMING
Elmira, New York

SOMETIMES WE TEACHERS feel so buffeted about by the sea of change that surrounds education that we close our ears to all suggestions—both good and bad. Methods of teaching have been altered so much in the past twenty years that even the best informed teacher has difficulty in selecting just what will be helpful in her own situation. Surrounded as we are by this constant aura of change many of us have clung to the ivy-covered methods of the past in the hope that, after all, that which is old is best.

The new teacher as well as the veteran may cling to past methods as the straw of hope. Two years ago when I started looking for my first teaching position I was shocked to hear a superintendent say, "You know, we have a tough time making our parents see that we're teaching arithmetic when we let our third graders play store." Though I didn't tell the superintendent my true feelings, I was in agreement with the parents who could not see the mathematical value of playing store. After all, I reasoned, my childhood third grade recollections contained no memory of playing anything and I had still managed to learn arithmetic.

Basing arithmetic learning on the child's classroom experiences is a new and old way of teaching. No kindergarten or first grade teacher would introduce her children to arithmetic without first letting the child see and play with numbers. Continuing to use the experience method to develop mathematical concepts with children in grades 2–6 is an approach new to the last twenty years and one that still must gain acceptance among the majority of teachers.

My earlier reference to the interview with the school superintendent is indicative that I felt children in grades 2–6 could understand arithmetic without the aid of experience situations. Having a little of the sulphur and molasses in my thinking, (what was good enough in my day is good enough now), I was convinced that methods used in my childhood had taught me arithmetic. And so I taught "carrying," "borrowing," and "the times tables," and "the goes intos" without benefit of a single "real" situation.

I might have continued in my negative attitude toward experience arithmetic had it not been for a summer shcool course in mathematics. Our instructor was wise enough to realize that in her class of forty-five people there were some minds such as mine that were unwilling to accept new teaching methods. Instead of lecturing to us on the need for experience before teaching arithmetic she let us see through demonstrations that some of our own adult mathematics thinking was unclear.

I remember one particularly vivid exercise entitled "counting backward from thirteen." Counting is a fundamental aspect of arithmetic. Many pre-school children can count so what could an adult possibly not understand about counting? This exercise was done as a demonstration. A volunteer started moving discs from a group of thirteen discs and counting backward as she moved the discs. One disc was moved—"Thirteen," she said, One more disc was moved, "Twelve," she said. "Show us twelve," said our smiling instructor. The last disc moved was held high and the voice of a kindergarten teacher startled us with, "That's not twelve—it's one." It was apparent that rather than counting with understanding, we had been simply saying the number names in reverse order. It disturbed and embarrassed me to think that I a third grade *teacher*, knew number names but not all the meanings contained in these

names. Perhaps I hadn't really understood arithmetic as well as I could have in my own school days. Maybe the advocates of experience arithmetic had a sound argument in insisting that *all* children must see and work with numbers in "real" situations before they could be expected to do pencil and paper arithmetic.

The demonstration of counting showed me how vague some of my own arithmetic concepts were. I pricked up my ears and opened my mind to this new teaching method. Two of the points that helped convert me to believing that experience arithmetic is a better way to teach are:

1. It makes use of natural learning patterns.
2. It clarifies situations which may otherwise leave no meaning for the child.

Experience arithmetic is the sensible way to teach because it makes use of natural learning patterns. Suppose you wish to learn to drive a car. How do you, an adult, begin your learning? You climb gingerly into a friend's or relative's car and while he instructs you start and, more than likely, stall his car. Eventually you chug off down the road of learning how to drive a car. It is only after many days or months of experience that you tackle the rules in the driver training manual in the hope that you too may one day have an operator's license. Did your learning need an experience basis? It most certainly did! And didn't you believe you were following the only common sense method of learning to drive?

Is it not reasonable to assume then that children would also need to follow these natural learning patterns whenever they attempt a new mathematical process such as multiplication? Handing children the tables and showing on the board how to multiply is like giving an adult the driver training manual and telling him to learn to drive a car. Neither method can insure any great degree of success because the normal learning pattern has been ignored. Experience arithmetic clarifies situations which may otherwise have little meaning for the child.

Sometimes our teaching resembles a meaningless "mumbo-jumbo" to children. To explain how mystifying our best arithmetic methods can be I must recall Bosco and Bozo. Bosco is a nine pound dog and Bozo is a four pound cat living within the pages of our arithmetic text. The pets are used in a series of situation problems. At one point children are asked to find the difference in the weights of the two animals. My teaching consisted of a thorough explanation of the difference in the sizes of the two pets and of the meaning of the key words "how much more." To me this was good teaching. True, my children had never had any experience in weighing, but I didn't think the experience was necessary because of my profuse explanation.

Can you imagine my frustration when the papers were in and over half my children had found the sum rather than the difference in the animal's weights?

Why hadn't my children understood this simple problem? Hadn't my explanation been clear? What was to bridge the gap between my words and the children's thinking? Did other teachers find themselves talking away precious minutes with a similar lack of results? How could children learn if we didn't tell them what we knew?

During the time my children were trying to solve the Bosco and Bozo problems I had no answer for these questions. Maybe the the actions of experience arithmetic will speak louder than all my "mumbo-jumbo" words!

EDITOR'S NOTE: Experiences with number and quantity in real situations give significance to learning and lift it from the traditional level of rote memorization. However, the final goal is to think and to work with numbers without reference to objects. We need experience with numbers just as we do with the quantities with which numbers are associated. Yes, there is a stage of working with things such as blocks and toys and there is also a necessary development beyond this stage. The wise teacher knows when to use various approaches to learning and when and how to provide the needed experience and practice. She will want her pupils finally to be able to think and to work with concepts, principles, and numbers found in the many uses of arithmetic and to do this with reasonable dispatch.